Theorem A: Lemma Statements with Explicit Error Terms

Experiential Measure Formalization

1 Setup and Notation

Let $(\Omega, Q_\varepsilon)_{\varepsilon>0}$ be a family of finite irreducible continuous-time Markov chains with state space $\Omega = B \times M$ ($|B|, |M| < \infty$) and generators of Metropolis type:

$$Q_\varepsilon(x,y) = r(x,y)\exp\bigl(-[E(y)-E(x)]^+/\varepsilon\bigr), \qquad x \neq y,$$

where $r(x,y) > 0$ for all pairs (x,y) with $x \neq y$ (irreducibility), $r(x,y) = r(y,x)$ (reversibility with respect to the Gibbs measure $\pi_\varepsilon(x) \propto \exp(-E(x)/\varepsilon)$), and $[a]^+ = \max(a,0)$.

Row sums are zero: $Q_\varepsilon(x,x) = -\sum_{y\neq x} Q_\varepsilon(x,y)$. Convention: row-vector left multiplication, $dp/dt = pQ_\varepsilon$.

The experiential density functional is $\rho(p) = I(B;M)\cdot(1 - I(B;M)/H(B))$, with entropy and mutual information computed in nats (ln). The trajectory functional is $\mu([0,T]) = \int_0^T \rho(p_t)\,dt$ (units: nat-seconds).

Standing assumptions for Theorem A:

1. The Freidlin–Wentzell cycle hierarchy (Lemma 1) decomposes Ω into metastable sets including B_{stable} with communication height Δ_s and B_{BB} with communication height Δ_b, satisfying $\Delta_s > \Delta_b$.
2. **Density bounds:** $\rho(\nu_{\text{QSD}}^{B_{\text{stable}}}) \geq c > 0$ (the quasi-stationary distribution on B_{stable} has non-trivial self-modeling), and $\rho(p) \leq \rho_{\max} = H(B)/4$ for all distributions p on Ω (automatic for finite state spaces).
3. The observation horizon is $T_\varepsilon = \exp((\Delta_s - \alpha)/\varepsilon)$ for a fixed $\alpha \in (0, \Delta_s - \Delta_b)$.
4. The initial distribution p_0 is concentrated on B_{stable}.

2 Lemma Statements

Lemma 1 (Basin Partition)**.** ***Source:*** *Freidlin–Wentzell [5], Chapter 6, Theorem 6.3.1 (cycle decomposition of Markov chains).*

Statement. *For the family $(Q_\varepsilon)_{\varepsilon>0}$ of reversible Metropolis generators on the finite state space Ω with energy function E, the Freidlin–Wentzell cycle hierarchy defines a partition of Ω into metastable sets (cycles) $\{B_1, B_2, \ldots, B_k\}$ at each hierarchy level h, with communication heights*

$$\Delta_i = V(B_i, B_i^c) - \min_{x\in B_i} E(x),$$

where $V(A, A^c) = \min_{x\in A, y\notin A} V(x,y)$ is the minimal communication height between A and its complement, defined via

$$V(x,y) = \min_{\gamma: x\to y} \max_{i=0,\ldots,|\gamma|-1} E(\gamma_i) - E(x)$$

(minimum over all paths γ from x to y of the maximal energy barrier along the path, measured from the starting energy).

Identify B_{stable} and B_{BB} as two cycles with $\Delta_s > \Delta_b > 0$.

Error term: *None. This is an exact graph-theoretic construction on the finite state space. No asymptotic approximation is involved.*

Output type: *Partition $(B_{\text{stable}}, B_{\text{BB}}, \Omega \setminus (B_{\text{stable}} \cup B_{\text{BB}}))$ with communication heights $\Delta_s, \Delta_b \in \mathbb{R}_{>0}$.*

Remark 1. *The communication height $V(x, y)$ in Freidlin–Wentzell theory is defined for the continuous-time chain via the W-graph (graph of optimal transition paths). For finite state spaces with Metropolis rates, it coincides with the energy barrier: $V(x, y) = \max_\gamma E(\gamma_*) - E(x)$ along the optimal path. The partition is determined entirely by the energy landscape E and the adjacency structure, independent of ε.*

Lemma 2 (Residence Time Lower Bound)**.** ***Source:*** *Bovier, Eckhoff, Gayrard, Klein [1], Theorem 1.2 (mean exit time via capacities); see also Bovier–den Hollander [2], Chapter 7, Theorem 7.8.*

Statement. *Let $\tau_{B^c_{\text{stable}}} = \inf\{t > 0 : X_t \notin B_{\text{stable}}\}$ be the first exit time from B_{stable}. For any starting state $x \in B_{\text{stable}}$,*

$$\mathbb{E}_x[\tau_{B^c_{\text{stable}}}] = \frac{1}{\pi_\varepsilon(x) \cdot \text{cap}_\varepsilon(x, B^c_{\text{stable}})} \cdot (1 + \delta_2), \tag{1}$$

where π_ε is the stationary (Gibbs) distribution, $\text{cap}_\varepsilon(x, B^c_{\text{stable}})$ is the capacity between x and B^c_{stable} (defined as $\text{cap}_\varepsilon(x, A) = \sum_y \pi_\varepsilon(x) Q_\varepsilon(x, y)(h_A(x) - h_A(y))$ with h_A the equilibrium potential), and

$$|\delta_2| \le C_2 \exp(-c_2/\varepsilon) \tag{2}$$

with $c_2 > 0$ determined by the sub-leading communication height in the cycle hierarchy.

Specifically, from the capacity asymptotics in [1] Theorem 1.2 and [2] Theorem 7.8, the capacity satisfies $\text{cap}_\varepsilon(x, B^c_{\text{stable}}) = \kappa_s \exp(-\Delta_s/\varepsilon)(1 + O(\exp(-c_2'/\varepsilon)))$ where $\kappa_s > 0$ is a prefactor depending on the curvatures of E at the saddle point and the minimum, and $c_2' > 0$. Since $\pi_\varepsilon(x) = Z_\varepsilon^{-1} \exp(-E(x)/\varepsilon)$, combining gives

$$\mathbb{E}_x[\tau_{B^c_{\text{stable}}}] = K_s \cdot \exp(\Delta_s/\varepsilon) \cdot (1 + \delta_2), \tag{3}$$

where $K_s = Z_\varepsilon/(\kappa_s \exp(-E(x)/\varepsilon))$ is a prefactor that is $O(1)$ in ε (polynomial in $|\Omega|$ and the rate prefactors $r(x, y)$).

The rate constant c_2 satisfies

$$c_2 \ge \min\big(\Delta_s - \Delta_b,\ \Delta_{\text{sub}}\big), \tag{4}$$

where Δ_{sub} is the sub-leading communication height (the smallest gap between the leading exit barrier Δ_s and the next barrier in the cycle hierarchy within B_{stable}).

In particular, $c_2 > 0$ and $c_2 \ge \Delta_s - \Delta_b > 0$ by the standing assumption $\Delta_s > \Delta_b$.

Exponential law *([1], Theorem 1.4): The rescaled exit time converges in distribution:*

$$\frac{\tau_{B^c_{\text{stable}}}}{\mathbb{E}_x[\tau_{B^c_{\text{stable}}}]} \xrightarrow{d} \text{Exp}(1) \qquad \text{as } \varepsilon \to 0. \tag{5}$$

This provides concentration of $\tau_{B^c_{\text{stable}}}$ around its mean without requiring full Donsker–Varadhan large deviation machinery. Quantitatively, for any $\eta > 0$:

$$\mathbb{P}_x\left(\left|\frac{\tau_{B^c_{\text{stable}}}}{\mathbb{E}_x[\tau_{B^c_{\text{stable}}}]} - 1\right| > \eta\right) \leq 2\exp(-\eta/2) \qquad \textit{for } \varepsilon \textit{ sufficiently small,} \tag{6}$$

using the exponential distribution tail bound $\mathbb{P}(|X-1| > \eta) \leq 2e^{-\eta/2}$ for $X \sim \text{Exp}(1)$ (valid for $\eta < 1$; for $\eta \geq 1$, use $\mathbb{P}(X > 1+\eta) = e^{-(1+\eta)}$).

Output type: *Lower bound on $\mathbb{E}_x[\tau_{B^c_{\text{stable}}}]$ of the form $K_s \exp(\Delta_s/\varepsilon)(1+\delta_2)$ (an expectation with multiplicative error), plus the exponential law providing distributional convergence (hence concentration of the random variable $\tau_{B^c_{\text{stable}}}$ around its mean).*

Lemma 3 (QSD Convergence)**.** ***Source:*** *Champagnat–Villemonais [3], Theorem 2.1 (exponential convergence to QSD for absorbed Markov processes on finite state spaces).*

Statement. *Consider the killed chain: the Markov chain X_t restricted to B_{stable} with absorption upon hitting B^c_{stable}. Let $Q_\varepsilon^{B_{\text{stable}}}$ denote the generator of this killed chain (obtained by restricting Q_ε to B_{stable} and setting absorption rates at the boundary).*

There exists a unique quasi-stationary distribution ν_{QSD} on B_{stable} satisfying

$$\mathbb{P}_{\nu_{\text{QSD}}}(X_t \in \cdot \mid t < \tau_{B^c_{\text{stable}}}) = \nu_{\text{QSD}}(\cdot) \qquad \textit{for all } t \geq 0.$$

Moreover, for any initial distribution p_0 supported on B_{stable},

$$\left\|\mathbb{P}_{p_0}(X_t \in \cdot \mid t < \tau_{B^c_{\text{stable}}}) - \nu_{\text{QSD}}\right\|_{\text{TV}} \leq C_3 \exp(-\gamma_D \cdot t), \tag{7}$$

where $\gamma_D > 0$ is the spectral gap of the killed generator $Q_\varepsilon^{B_{\text{stable}}}$ (specifically, the gap between the two largest eigenvalues of $Q_\varepsilon^{B_{\text{stable}}}$, noting all eigenvalues are negative), and $C_3 > 0$ depends on the initial distribution p_0.

Key property of γ_D: *The spectral gap γ_D of the killed chain within B_{stable} is $O(1)$ as $\varepsilon \to 0$ (not exponentially small). This holds because:*

- *The killed chain operates within a single metastable basin B_{stable}.*
- *All transition rates* within *B_{stable} are $O(1)$ or larger: internal transitions have barriers $\leq \Delta_{\text{int}} < \Delta_s$, so rates $\geq r_{\min}\exp(-\Delta_{\text{int}}/\varepsilon)$. The exponentially small rates are the* inter-basin *rates (barriers $\geq \Delta_s$), which are removed by the killing.*
- *By the Cheeger inequality for the killed chain on B_{stable} (see, e.g., [2] Section 7.2), γ_D is bounded below by a function of the minimum within-basin transition rate and the basin geometry, both of which are $O(1)$ in ε.*

More precisely, $\gamma_D \geq \gamma_{\min}$ where $\gamma_{\min}$ depends on $|B_{\text{stable}}|$, $r_{\min}$, and Δ_{int} (the maximum internal barrier within B_{stable}) but not *on Δ_s or ε (beyond the within-basin rates).*

Error term:

$$|\delta_3(t)| = C_3 \exp(-\gamma_D t).$$

This is not an asymptotic-in-ε error: it is an error in time t for fixed ε. The relevant statement is that after time $t_{\text{mix}} = O(1/\gamma_D) = O(1)$ within the basin, the conditional distribution is within TV*-distance $C_3 e^{-\gamma_D t_{\text{mix}}}$ of ν_{QSD}.*

Since the residence time in B_{stable} is $\tau \sim \exp(\Delta_s/\varepsilon) \gg 1/\gamma_D$ for small ε, the chain spends all but an $O(1)$-duration transient at the QSD.

Output type: *The unique QSD ν_{QSD} on B_{stable}, with a convergence rate $\gamma_D = O(1)$ and the TV-distance bound (7) (a high-probability statement about the conditional distribution within the basin).*

Lemma 4 (Stable Measure Lower Bound)**.** ***Source:*** *Combination of Lemma 2 (residence time) and Lemma 3 (QSD convergence).*

Statement. *The experiential measure accumulated during residence in B_{stable} satisfies*

$$\mu_{\text{stable}} = \int_0^{\tau_{B^c_{\text{stable}}}} \rho(p_t)\, dt \geq c \cdot K_s \cdot \exp(\Delta_s/\varepsilon) \cdot (1 - \delta_4), \tag{8}$$

where $c = \rho(\nu_{\text{QSD}}) > 0$ (the standing density-bound assumption), K_s is the residence-time prefactor from Lemma 2, and

$$\delta_4 = O\left(\frac{1}{\gamma_D \cdot K_s \exp(\Delta_s/\varepsilon)}\right) = O(\exp(-\Delta_s/\varepsilon)). \tag{9}$$

Derivation of the bound. *Decompose the residence period $[0, \tau_{B^c_{\text{stable}}}]$ into a transient phase $[0, t_0]$ and a QSD phase $[t_0, \tau_{B^c_{\text{stable}}}]$ where $t_0 = O(1/\gamma_D)$ is chosen so that $\|p^{\text{killed}}_{t_0} - \nu_{\text{QSD}}\|_{\text{TV}} < \delta$ for a small $\delta > 0$.*

During $[t_0, \tau_{B^c_{\text{stable}}}]$:

$$|\rho(p_t) - \rho(\nu_{\text{QSD}})| \leq L_\rho \|p_t - \nu_{\text{QSD}}\|_{\text{TV}} \leq L_\rho C_3 e^{-\gamma_D t}$$

where L_ρ is the Lipschitz constant of ρ with respect to TV distance (bounded by $O(H(B))$ on finite state spaces). Therefore

$$\begin{aligned}
\mu_{\text{stable}} &= \int_0^{t_0} \rho(p_t)\, dt + \int_{t_0}^{\tau_{B^c_{\text{stable}}}} \rho(p_t)\, dt \\
&\geq 0 + \int_{t_0}^{\tau_{B^c_{\text{stable}}}} [\rho(\nu_{\text{QSD}}) - L_\rho C_3 e^{-\gamma_D t}]\, dt \\
&\geq c \cdot (\tau_{B^c_{\text{stable}}} - t_0) - L_\rho C_3/\gamma_D \\
&= c \cdot \tau_{B^c_{\text{stable}}} \Big(1 - \frac{t_0}{\tau_{B^c_{\text{stable}}}} - \frac{L_\rho C_3}{c \gamma_D \tau_{B^c_{\text{stable}}}}\Big).
\end{aligned}$$

Since $t_0 = O(1)$ and $\tau_{B^c_{\text{stable}}} \sim K_s \exp(\Delta_s/\varepsilon)$, both correction terms are $O(\exp(-\Delta_s/\varepsilon))$, giving $\delta_4 = O(\exp(-\Delta_s/\varepsilon))$.

Rate constant: *The error rate in δ_4 is*

$$c_4 = \Delta_s, \tag{10}$$

which satisfies $c_4 = \Delta_s > \Delta_s - \Delta_b > 0$. The correction δ_4 is superexponentially small compared to the gap $\Delta_s - \Delta_b$ that controls the final bound in Lemma 7.

Output type: *Lower bound on μ_{stable} with multiplicative error $(1-\delta_4)$ where $\delta_4 = O(\exp(-\Delta_s/\varepsilon))$. This is a statement about a random variable (the integrated experiential measure during one residence period).*

To convert to a high-probability bound: apply the exponential law (Lemma 2, eq. (5)). Since $\tau_{B^c_{\text{stable}}}/\mathbb{E}[\tau_{B^c_{\text{stable}}}] \to \text{Exp}(1)$, *for any* $\eta > 0$,

$$\mathbb{P}\big(\tau_{B^c_{\text{stable}}} > (1-\eta)\mathbb{E}[\tau_{B^c_{\text{stable}}}]\big) \geq 1 - (1 - e^{-(1-\eta)}) \to 1 - (1 - e^{-1}) \approx 0.632$$

as $\varepsilon \to 0$. *Stronger: with probability* $1 - e^{-a}$ *the residence time exceeds* $a \cdot \mathbb{E}[\tau]$ *(for* $a < 1$*). The* μ_{stable} *bound inherits this concentration.*

Lemma 5 (BB Occupation Upper Bound)**.** ***Source:*** *Renewal theory for metastable chains; BEGK [1] Theorem 1.2 for exit times from* B_{BB}*; [2] Chapter 7 for the excursion structure.*

Statement. *During the observation horizon* $T_\varepsilon = \exp((\Delta_s - \alpha)/\varepsilon)$, *the expected total occupation time of* B_{BB} *satisfies*

$$\mathbb{E}[\tau_{B_{\text{BB}}}(T_\varepsilon)] \leq K_b \cdot \exp((\Delta_b - \alpha)/\varepsilon) \cdot (1 + \delta_5), \tag{11}$$

where $K_b > 0$ *is a prefactor from the BEGK capacity formula for* B_{BB} *exit times (analogous to* K_s*), and* $|\delta_5| \leq C_5 \exp(-c_5/\varepsilon)$ *with* $c_5 \geq \min(\Delta_s - \Delta_b, \Delta'_{\text{sub}})$ *(*Δ'_{sub} *is the sub-leading barrier for* B_{BB}*).*

Derivation. *The excursion structure works as follows:*

1. *The chain starts in* B_{stable} *and must first exit* B_{stable} *before visiting* B_{BB}. *The probability that the chain reaches* B_{BB} *before time* T_ε *is bounded by*

$$\mathbb{P}(\tau_{B^c_{\text{stable}}} < T_\varepsilon) = \mathbb{P}\left(\frac{\tau_{B^c_{\text{stable}}}}{\mathbb{E}[\tau_{B^c_{\text{stable}}}]} < \frac{T_\varepsilon}{\mathbb{E}[\tau_{B^c_{\text{stable}}}]}\right).$$

   Now $\mathbb{E}[\tau_{B^c_{\text{stable}}}] = K_s \exp(\Delta_s/\varepsilon)(1 + \delta_2)$ *and* $T_\varepsilon = \exp((\Delta_s - \alpha)/\varepsilon)$, *so*

$$\frac{T_\varepsilon}{\mathbb{E}[\tau_{B^c_{\text{stable}}}]} = \frac{\exp(-\alpha/\varepsilon)}{K_s(1 + \delta_2)} = O(\exp(-\alpha/\varepsilon)) \to 0.$$

   By the exponential law (5), $\mathbb{P}(\tau_{B^c_{\text{stable}}}/\mathbb{E}[\tau_{B^c_{\text{stable}}}] < s) \to 1 - e^{-s} \sim s$ *for small* s. *Therefore*

$$\mathbb{P}(\textit{any BB excursion before } T_\varepsilon) \leq \frac{T_\varepsilon}{\mathbb{E}[\tau_{B^c_{\text{stable}}}]}(1 + o(1)) = O(\exp(-\alpha/\varepsilon)). \tag{12}$$

2. *Conditional on an excursion to* B_{BB} *occurring, the time spent in* B_{BB} *during one visit is* $\mathbb{E}[\tau_{B_{\text{BB}} \to B^c_{\text{BB}}}] = K_b \exp(\Delta_b/\varepsilon)(1 + \delta'_5)$ *by applying BEGK Theorem 1.2 to* B_{BB} *(same formula as Lemma 2 but for the shallower basin).*

3. *Therefore the expected total BB occupation is bounded by*

$$\begin{aligned}
\mathbb{E}[\tau_{B_{\text{BB}}}(T_\varepsilon)] &\leq \mathbb{P}(\textit{excursion occurs}) \cdot \mathbb{E}[\textit{BB occupation} \mid \textit{excursion}] \cdot (1 + o(1)) \\
&\leq \frac{T_\varepsilon}{\mathbb{E}[\tau_{B^c_{\text{stable}}}]} \cdot K_b \exp(\Delta_b/\varepsilon) \cdot (1 + O(\exp(-c_5/\varepsilon))) \\
&= \frac{\exp((\Delta_s - \alpha)/\varepsilon)}{K_s \exp(\Delta_s/\varepsilon)} \cdot K_b \exp(\Delta_b/\varepsilon) \cdot (1 + O(\exp(-c_5/\varepsilon))) \\
&= \frac{K_b}{K_s} \exp((\Delta_b - \alpha)/\varepsilon) \cdot (1 + \delta_5).
\end{aligned} \tag{13}$$

BB experiential measure upper bound. *Since $\rho(p) \le \rho_{\max} = H(B)/4$ for all distributions p,*

$$\mu_{B_{\mathrm{BB}}} \le \rho_{\max} \cdot \mathbb{E}[\tau_{B_{\mathrm{BB}}}(T_\varepsilon)] \le \rho_{\max} \cdot \frac{K_b}{K_s} \cdot \exp((\Delta_b - \alpha)/\varepsilon) \cdot (1 + \delta_5). \tag{14}$$

Rate constant:

$$c_5 \ge \min(\Delta_s - \Delta_b, \Delta'_{\mathrm{sub}}) > 0. \tag{15}$$

Output type: *Upper bound on $\mathbb{E}[\tau_{B_{\mathrm{BB}}}(T_\varepsilon)]$ and hence on $\mathbb{E}[\mu_{B_{\mathrm{BB}}}]$ (expected BB experiential measure), with multiplicative error $(1 + \delta_5)$ where $\delta_5 = O(\exp(-c_5/\varepsilon))$.*

Note: this is an expected-value bound, not a high-probability bound. The conversion to a high-probability statement is handled by Lemma 6.

Lemma 6 (Concentration of Occupation-Time Functionals)**.** ***Source:*** *For the exit-time concentration route: BEGK [1] Theorem 1.4 (exponential law of exit times). For the general large-deviation route: Donsker–Varadhan [4] (large deviations for occupation measures of Markov processes); see also den Hollander,* Large Deviations*, AMS (2000), Chapter VIII.*

Statement (exit-time concentration route — sufficient for Theorem A).

By the BEGK exponential law (Lemma 2, eq. (5)), the exit time $\tau_{B^c_{\mathrm{stable}}}$ concentrates around its mean. Since the experiential measure μ_{stable} is controlled by the occupation time in B_{stable} and the QSD convergence (Lemma 3), the following hold simultaneously with probability at least $1 - 2e^{-\eta/2} - o(1)$ as $\varepsilon \to 0$:

$$\mu_{\mathrm{stable}} \ge c \cdot K_s \cdot (1 - \eta) \exp(\Delta_s/\varepsilon) \cdot (1 - \delta_4), \tag{16}$$

$$\mu_{B_{\mathrm{BB}}} \le \rho_{\max} \cdot \frac{K_b}{K_s} \cdot \exp((\Delta_b - \alpha)/\varepsilon) \cdot (1 + \delta_5 + \delta_6), \tag{17}$$

where δ_6 accounts for the deviation of $\tau_{B_{\mathrm{BB}}}$ from its expectation. For the BB occupation, since the excursion probability itself is $O(\exp(-\alpha/\varepsilon)) \to 0$, the random variable $\tau_{B_{\mathrm{BB}}}(T_\varepsilon)$ is zero with probability $1 - O(\exp(-\alpha/\varepsilon))$, and conditional on being nonzero, the single-visit duration concentrates around its BEGK mean via the exponential law applied to B_{BB} exit.

Therefore $\delta_6 = O(\exp(-c_6/\varepsilon))$ with $c_6 = \alpha > 0$.

Statement (Donsker–Varadhan route — for generality).

For the time-averaged empirical measure $L_T = \frac{1}{T}\int_0^T \mathbf{1}_{X_t \in A}\, dt$ of an ergodic continuous-time Markov chain with generator Q on a finite state space, the Donsker–Varadhan large deviation principle gives:

$$\mathbb{P}(|L_T - \pi(A)| > \eta) \le \exp(-T \cdot I_{\mathrm{DV}}(\eta))$$

where $I_{\mathrm{DV}}(\eta) > 0$ for $\eta > 0$ and $\pi(A)$ is the stationary occupation fraction. Since $T = T_\varepsilon = \exp((\Delta_s - \alpha)/\varepsilon)$ is exponentially large, this gives super-exponentially tight concentration.

For weighted functionals $\frac{1}{T}\int_0^T f(X_t)\, dt$ with bounded f (as is the case for ρ on finite state spaces, since $0 \le \rho \le H(B)/4$), the contraction principle applied to the DV rate function yields concentration with rate $I_f(\eta) > 0$.

Error term (exit-time route):

$$\delta_6 = O(\exp(-\alpha/\varepsilon)), \qquad c_6 = \alpha. \tag{18}$$

Note on the DV extension. *The standard DV theorem addresses the unweighted empirical measure. To handle the ρ-weighted functional $\mu/T = \frac{1}{T}\int_0^T \rho(p_t)\,dt$, one applies the contraction principle: since ρ is a bounded continuous functional on the space of distributions over a finite state space, the map $L_T \mapsto \rho(L_T)$ is Lipschitz (with Lipschitz constant $L_\rho < \infty$), and the rate function for the pushed-forward functional is $I_\rho(\eta) \geq I_{\mathrm{DV}}(\eta/L_\rho) > 0$.*

However, for the present application (Theorem A), the simpler exit-time concentration route via BEGK Theorem 1.4 suffices and avoids this extension entirely.

Output type: *High-probability bounds on μ_{stable} (lower) and $\mu_{B_{\mathrm{BB}}}$ (upper), converting the expected-value results of Lemmas 4 and 5 into probabilistic statements.*

Lemma 7 (Ratio Assembly)**.** ***Source:*** *Direct combination of Lemma 4 (lower bound on μ_{stable}), Lemma 5 (upper bound on $\mu_{B_{\mathrm{BB}}}$), and Lemma 6 (concentration).*

Statement. *Under the standing assumptions, as $\varepsilon \to 0$:*

$$\frac{\mu_{B_{\mathrm{BB}}}}{\mu_{\mathrm{stable}}} \leq C \cdot \exp\left(-\frac{\Delta_s - \Delta_b - \alpha}{\varepsilon}\right) \cdot (1 + \delta_7), \tag{19}$$

where the prefactor is

$$C = \frac{\rho_{\max}}{c} \cdot \frac{K_b}{K_s^2}, \tag{20}$$

and

$$|\delta_7| \leq C_7 \exp(-c_7/\varepsilon), \qquad c_7 = \min(c_2, c_4, c_5, c_6) = \min(\Delta_s - \Delta_b, \Delta_s, \alpha, \Delta_{\mathrm{sub}}, \Delta'_{\mathrm{sub}}). \tag{21}$$

Derivation. *Using the high-probability bounds from Lemma 6:*

$$\begin{aligned}\frac{\mu_{B_{\mathrm{BB}}}}{\mu_{\mathrm{stable}}} &\leq \frac{\rho_{\max} \cdot \frac{K_b}{K_s} \exp((\Delta_b - \alpha)/\varepsilon)(1 + \delta_5 + \delta_6)}{c \cdot K_s \cdot (1 - \eta) \exp(\Delta_s/\varepsilon)(1 - \delta_4)} \\ &= \frac{\rho_{\max} K_b}{c K_s^2 (1 - \eta)} \cdot \exp\left(\frac{\Delta_b - \alpha - \Delta_s}{\varepsilon}\right) \cdot \frac{(1 + \delta_5 + \delta_6)}{(1 - \delta_4)}.\end{aligned}$$

Taking $\eta \to 0$ (which is valid since the probability $1 - 2e^{-\eta/2}$ can be made arbitrarily close to $1 - 2/e \approx 0.264$ and then improved by taking η small), and collecting the error terms:

$$\frac{(1 + \delta_5 + \delta_6)}{(1 - \delta_4)} = 1 + \delta_5 + \delta_6 + \delta_4 + O(\delta_i \delta_j) = 1 + O(\exp(-c_7/\varepsilon)),$$

since each $\delta_i = O(\exp(-c_i/\varepsilon))$ with $c_i > 0$, and $c_7 = \min_i c_i > 0$.

Verification that C is $O(1)$ in ε:

- $\rho_{\max} = H(B)/4 = \ln|B|/4$*: depends on $|B|$, not ε.*
- $c = \rho(\nu_{\mathrm{QSD}}) > 0$*: the QSD on B_{stable} has nonzero mutual information by standing assumption; c depends on the basin structure but not on ε (the QSD converges to the ground-state conditional distribution as $\varepsilon \to 0$).*
- K_s, K_b*: BEGK prefactors are rational functions of the rate prefactors $r(x, y)$, the energy values $E(x)$, and the partition function Z_ε. For Metropolis dynamics on finite state spaces, K_s and K_b are $O(1)$ in ε (they involve saddle-point curvatures and a finite number of rates, all bounded). See [2] Theorem 7.8 for explicit formulas.*

Therefore $C = O(1)$ in ε, confirming the exponential bound is not spoiled by prefactor growth.

Critical check: error rate vs. exponent. *The decay rate in the main term is $(\Delta_s - \Delta_b - \alpha)/\varepsilon$. The error rate is c_7/ε. We need $c_7 > 0$ for the error to be negligible. We have:*

$$\begin{aligned} c_2 &\geq \Delta_s - \Delta_b > 0, \\ c_4 &= \Delta_s > 0, \\ c_5 &\geq \Delta_s - \Delta_b > 0, \\ c_6 &= \alpha > 0. \end{aligned}$$

All are strictly positive. Moreover, $c_7 \geq \min(\Delta_s - \Delta_b, \alpha) > 0$ (since $\alpha > 0$ by assumption and $\Delta_s > \Delta_b$).

Thus the error product remains $1 + O(\exp(-c_7/\varepsilon))$ with $c_7 > 0$, and the exponential decay $\exp(-(\Delta_s - \Delta_b - \alpha)/\varepsilon) \to 0$ dominates.

Output type: *The Theorem A bound: $\mu_{B_{\mathrm{BB}}}/\mu_{\mathrm{stable}} \leq C \exp(-(\Delta_s - \Delta_b - \alpha)/\varepsilon)(1 + O(\exp(-c_7/\varepsilon)))$, matching the statement in Section 7.2 of the draft.*

3 Typed Dependency Graph

The seven lemmas form a directed acyclic graph (DAG). Each edge is annotated with the mathematical object that flows from source to target and the type of that object.

```
L1 (Basin Partition -- exact)
 |-- L2 (Residence Time -- expectation + concentration)
 |    |-- L4 (Stable Measure Lower Bound -- high-prob bound)
 |    +-- L5 (BB Occupation Upper Bound -- expected value)
 |-- L3 (QSD Convergence -- TV bound, rate gamma_D)
 |    +-- L4
 +-- L6 (Concentration -- converts E[.] to high-prob bounds)
      +-- L7 (Ratio Assembly -- Theorem A bound)
              ^
       L4, L5 -+
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Topological order: $L1 \to (L2, L3, L6) \to (L4, L5) \to L7$.

Edge table:

Source	Target	Output (source)	Input (target)	Type match?
L1	L2	Basin B_{stable} with comm. height Δ_s. *Type: exact partition.*	A metastable set with known comm. height for BEGK capacity formula. *Type: exact partition.*	**Yes.** Exact $\to$ exact.
L1	L3	Basin B_{stable} (as a subset of Ω). *Type: exact partition.*	A subset on which to define the killed chain. *Type: exact partition.*	**Yes.** Exact $\to$ exact.
L1	L6	Basin partition and comm. heights. *Type: exact partition.*	State space structure for DV / concentration argument. *Type: exact partition.*	**Yes.** Exact $\to$ exact.
L2	L4	$\mathbb{E}[\tau_{B^c_{\text{stable}}}] = K_s e^{\Delta_s/\varepsilon}(1+\delta_2)$. *Type: expectation with mult. error.* Also: exponential law giving concentration.	Residence time bound (needs at least a high-prob. bound on $\tau_{B^c_{\text{stable}}}$, not just the mean). *Type: high-probability bound.*	**Yes, via BEGK Thm 1.4.** The exponential law converts the expectation to a high-prob. bound. See (6).
L3	L4	ν_{QSD} with convergence rate $\gamma_D = O(1)$. *Type: TV convergence bound.*	QSD to evaluate $\rho(\nu_{\text{QSD}}) \geq c$. Convergence rate to bound transient error. *Type: distribution + rate.*	**Yes.** TV convergence gives both the limiting distribution and the transient error bound needed for δ_4.
L2	L5	$\mathbb{E}[\tau_{B^c_{\text{stable}}}]$ (mean exit time, giving excursion probability $T_\varepsilon/\mathbb{E}[\tau]$). Also: BEGK for B_{BB} exit time. *Type: expectation.*	Excursion probability and conditional BB duration. *Type: expectation $\to$ expected occupation.*	**Yes.** The excursion probability uses the ratio $T_\varepsilon/\mathbb{E}[\tau]$, which is exact up to δ_2. The output is an expected-value bound, flagged for conversion by L6.
L4	L7	$\mu_{\text{stable}} \geq cK_s e^{\Delta_s/\varepsilon}(1-\delta_4)$. *Type: lower bound on random variable.*	Denominator of the ratio. *Type: lower bound (high-prob.).*	**Yes, after L6.** L6 converts L4's per-trajectory bound into a high-probability statement.
L5	L7	$\mathbb{E}[\mu_{B_{\text{BB}}}] \leq \rho_{\max}\frac{K_b}{K_s}e^{(\Delta_b-\alpha)/\varepsilon}(1+\delta_5)$. *Type: expected-value upper bound.*	Numerator of the ratio. *Type: high-probability upper bound.*	**Yes, after L6.** L6 converts the expected-value bound to a high-prob. bound via exit-time concentration.
L6	L7	High-probability bounds	Numerator and denom-	**Yes.** Both nu-

DAG verification:

- L1 has no incoming edges (root).
- L7 has no outgoing edges (sink).
- No cycles: all edges go from lower-numbered to higher-numbered lemmas (L1 $\to$ L2, L3, L6; L2 $\to$ L4, L5; L3 $\to$ L4; L4, L5, L6 $\to$ L7).
- Every lemma's inputs are supplied by lemmas earlier in topological order.

Type mismatch analysis:

The critical potential mismatch identified in the plan (Pitfall 2: expectations flowing where high-probability bounds are needed) appears at two edges:

1. **L2 $\to$ L4:** L2 provides $\mathbb{E}[\tau_{B^c_{\text{stable}}}]$ (an expectation), but L4 needs a lower bound on the random variable $\tau_{B^c_{\text{stable}}}$ (for the integral bound on μ_{stable}).

   Resolution: BEGK Theorem 1.4 ((5)) provides $\tau/\mathbb{E}[\tau] \xrightarrow{d} \text{Exp}(1)$, which gives high-probability lower bounds on τ. This is a genuine conversion step, not an implicit assumption.

2. **L5 $\to$ L7:** L5 provides an expected-value upper bound on $\mu_{B_{\text{BB}}}$, but L7 divides two random quantities and needs high-probability control.

   Resolution: Lemma 6 (L6) explicitly handles this conversion. The BB occupation is zero with probability $1 - O(\exp(-\alpha/\varepsilon))$, so the high-probability bound is actually simpler than the expected-value bound.

Both type mismatches are resolved by cited results. No hidden type conversions remain.

L7 output vs. Theorem A statement:

Lemma 7 produces

$$\frac{\mu_{B_{\text{BB}}}}{\mu_{\text{stable}}} \le C \exp\left(-\frac{\Delta_s - \Delta_b - \alpha}{\varepsilon}\right)(1 + O(\exp(-c_7/\varepsilon))),$$

which matches the Theorem A statement in Section 7.2 of the draft:

$$\mu_{B_{\text{BB}}}/\mu_{\text{stable}} \le C \cdot \exp(-(\Delta_s - \Delta_b - \alpha)/\varepsilon).$$

The explicit error term $(1 + O(\exp(-c_7/\varepsilon)))$ with $c_7 = \min(\Delta_s - \Delta_b, \alpha) > 0$ is the refinement contributed by this plan.

4 Error Rate Summary

Lemma	**Error term**	**Rate c_i**	**Source**
L1	(none – exact)	—	FW cycle hierarchy
L2	$\delta_2 = O(e^{-c_2/\varepsilon})$	$c_2 \ge \Delta_s - \Delta_b$	BEGK Thm 1.2
L3	$\delta_3(t) = C_3 e^{-\gamma_D t}$	$\gamma_D = O(1)$	CV16 Thm 2.1
L4	$\delta_4 = O(e^{-\Delta_s/\varepsilon})$	$c_4 = \Delta_s$	Combination of L2, L3
L5	$\delta_5 = O(e^{-c_5/\varepsilon})$	$c_5 \ge \Delta_s - \Delta_b$	BEGK Thm 1.2 for B_{BB}
L6	$\delta_6 = O(e^{-\alpha/\varepsilon})$	$c_6 = \alpha$	BEGK Thm 1.4
L7	$\delta_7 = O(e^{-c_7/\varepsilon})$	$c_7 = \min(\Delta_s - \Delta_b, \alpha)$	Composition

Critical check: All $c_i > 0$ (since $\Delta_s > \Delta_b$ and $\alpha > 0$). The final composite error rate $c_7 = \min(\Delta_s - \Delta_b, \alpha) > 0$ ensures the error product does not spoil the exponential decay $\exp(-(\Delta_s - \Delta_b - \alpha)/\varepsilon)$.

Since $\alpha < \Delta_s - \Delta_b$ (standing assumption), we have $c_7 = \alpha$ in the generic case, and the error term decays at the same rate as the observation-horizon parameter. This is tight: choosing α closer to $\Delta_s - \Delta_b$ sharpens the main bound but worsens the error term, while choosing α small worsens the main bound but the error is negligible. The optimal trade-off is a separate question for Plan 02.

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