

# Chirality from $h_3(\mathbb{O})$ : Standard Model Gauge Group and Chiral Representation from Self-Modeling via Exceptional Jordan Algebra

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## Abstract

Self-modeling—the requirement that a physical system contains a faithful model of itself—forces quantum mechanics with  $M_n(\mathbb{C})^{sa}$  state spaces [1]. We argue that the exceptional Jordan algebra  $h_3(\mathbb{O})$  is the unique candidate for the “universe algebra” via its non-composability (it cannot be a subsystem of a larger system). A  $C^*$ -observer probing  $h_3(\mathbb{O})$  induces a Peirce decomposition under a rank-1 idempotent  $E_{11}$ . We prove that the  $C^*$ -algebra continuous functional calculus, applied to the Peirce operators—which have negative eigenvalues—yields complex coefficients, extending the measurement algebra from  $M_{16}(\mathbb{R})$  to  $M_{16}(\mathbb{C})$  and upgrading the symmetry from  $\text{Spin}(9)$  to  $\text{Spin}(10)$  acting on a 16-dimensional complex Weyl spinor  $S_{10}^+$ . A single additional algebraic choice—a complex structure  $u \in S^6 \subset \text{Im}(\mathbb{O})$ —simultaneously determines both the Standard Model gauge group  $\text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)_Y$  via  $F_4$  intersection and the chiral (left-handed) fermion representation via the  $\text{Cl}(6)$  volume form. The result is conditional on one input beyond the self-modeling axioms: the identification of  $h_3(\mathbb{O})$  via non-composability. The observer’s choice of idempotent and complex structure are instances of necessary symmetry reduction—algebraically required choices within conjugacy classes whose direction is physically irrelevant. The complexification step, previously assumed, is traced to the *combination* of the observer’s  $C^*$ -nature (Paper 5) and the negative-spectrum Peirce operators of  $h_3(\mathbb{O})$ : neither input alone suffices. We present the 9-link chain from self-modeling axioms to a one-generation chiral Standard Model representation, with explicit classification of each link as proved, necessary symmetry reduction, or an open gap.

## 1 Introduction

The Standard Model gauge group  $\text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)_Y$  with its chiral fermion representation—left-handed doublets and right-handed singlets under  $\text{SU}(2)_L$ —is the empirical target of any program that seeks to derive particle physics from deeper principles. In a companion paper [1], we showed that *self-modeling*—the requirement that a physical system contains a faithful model of itself (a subsystem whose states separate the states of the whole), updated

through its boundary—forces quantum mechanics: the state space of any self-modeling system is isomorphic to  $M_n(\mathbb{C})^{sa}$ , the self-adjoint part of a matrix C\*-algebra.

This result leaves open a structural question. Simple  $M_n(\mathbb{C})$  algebras produce the gauge group  $U(n)$  via the spectral triple construction of Connes [2]—not the Standard Model. To obtain the SM gauge group within the Connes–Chamseddine–Marcolli framework, the finite algebra must be taken as a direct sum,  $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ , an input chosen to match observation. This is a *structural obstruction*: self-modeling forces simple matrix algebras, but simple matrix algebras cannot reproduce the SM within the spectral triple approach. The obstruction is not a computational gap but a no-go result [3]: no Dirac operator on  $M_n(\mathbb{C})$  yields the SM gauge group.

This motivates turning to a different algebraic structure: the exceptional Jordan algebra  $h_3(\mathbb{O})$ , the algebra of  $3 \times 3$  Hermitian matrices over the octonions  $\mathbb{O}$ . The exceptional Jordan algebra occupies a singular position in the Jordan–von Neumann–Wigner classification [4]: it is the unique *simple* finite-dimensional formally real Jordan algebra that is *non-composable*—it cannot participate in any composite with a nontrivial system [5]. Under Level IV mathematical realism, if an algebra can form composites then composites of it exist, making it a subsystem of a larger structure. The universe, by definition, is not a subsystem; therefore its algebra must be non-composable. This makes  $h_3(\mathbb{O})$  the unique candidate for the “universe algebra” whose internal structure should encode the particle content [6, 7].

The connection between  $h_3(\mathbb{O})$  and the Standard Model has been explored from several directions. Todorov and Drenska [8] showed that the  $F_4 = \text{Aut}(h_3(\mathbb{O}))$  intersection with the stabilizer of a rank-1 idempotent gives the SM gauge group. Furey [9] showed that the Clifford algebra  $\text{Cl}(6)$ , via the Witt decomposition, produces automatic chirality for a single generation of SM fermions. Boyle [10] studied the complexification  $h_3^{\mathbb{C}}(\mathbb{O})$  and its connection to  $E_6$ . Krasnov [11] explored the role of pure spinors and complex structures on  $\mathbb{O}^2$ .

The present paper contributes four new results to this program:

1. **Complexification traced to C\*-observer nature.** We prove that the complexification  $\text{Spin}(9) \rightarrow \text{Spin}(10)$  is produced by the *combination* of the observer’s C\*-algebra nature (from self-modeling [1]) and the negative-spectrum Peirce operators of  $h_3(\mathbb{O})$ : the continuous functional calculus of the observer’s C\*-algebra, applied to these operators, yields complex coefficients, extending the measurement algebra from  $M_{16}(\mathbb{R})$  to  $M_{16}(\mathbb{C})$  (Theorem 7). Neither input alone suffices—a real-algebraic observer would have no complex functional calculus; an operator with non-negative spectrum would have a real square root. This replaces the bare assumption of complexification in [10] with a conditional derivation from identified inputs. This is Part A of the argument (Sec. 2).
2. **One choice, two consequences.** The  $F_4$  intersection route (Todorov–Drenska) and the  $\text{Cl}(6)/\text{Pati-Salam}$  route (Furey) are traced to a *single* algebraic input: the choice of a complex structure  $u \in S^6 \subset \text{Im}(\mathbb{O})$ . Both routes produce the same gauge algebra  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ ; the  $\text{Cl}(6)$  route additionally provides the chiral representation.
3. **Symmetry reductions shown to be algebraically necessary.** The two symmetry-reducing steps ( $E_{11}$  and  $u$ ) are shown to be instances of *necessary symmetry reduction*: the observer must occupy an idempotent and must complexify non-equivariantly, while

Table 1: The 9-link derivation chain from self-modeling to the chiral Standard Model representation. Status: **Proved** = new result derived in this work or companion papers; **Proved (standard)** = follows from established representation theory once the prior link is granted (the cited literature provides the computation); **Gap** = a conditioning input not derived from self-modeling; *NSR* = necessary symmetry reduction (direction is gauge).

Link	Statement
L1	Self-modeling forces $M_n(\mathbb{C})^{sa}$ (C*-algebra quantum mechanics)
L2	Non-composability of $h_3(\mathbb{O})$ identifies it as the unique “universe algebra”
L3	Observer occupies rank-1 idempotent $E_{11}$ ; Peirce decomposition gives $V_1(1) \oplus V_{1/2}(16) \oplus V_0(10)$ , <i>NSR</i>
L4	C*-observer functional calculus yields complexification (Theorem 7): $V_{1/2} \otimes_{\mathbb{R}} \mathbb{C} = S_{10}^+$ , $\text{Spin}(9) \rightarrow \text{Spin}(10)$
L5	Complexification upgrades $F_4 \rightarrow E_6$ , $\mathbf{27} \rightarrow \mathbf{1} \oplus \mathbf{10} \oplus \mathbf{16}$ under $\text{Spin}(10)$
L6	Observer selects complex structure $u \in S^6 \subset \text{Im}(\mathbb{O})$ ; $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$
L7	$u$ defines $\text{Cl}(6) \subset \text{Cl}(10)$ ; volume form $\omega_6$ selects LEFT embedding: $\mathbf{16} \rightarrow (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$
L8	Same $u$ breaks $F_4 \supset [\text{SU}(3)_C \times \text{SU}(3)_J]/\mathbb{Z}_3$ ; intersection with $\text{Spin}(9)$ gives $\text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)$
L9	$\text{Cl}(6)$ /Pati-Salam route gives <b>same</b> SM gauge group as $F_4$ intersection, <b>plus</b> chiral representation

all choices are conjugate. This reduces the conditioning inputs from three to one (Gap A).

4. **Complete chain with explicit gaps.** No prior work connects the self-modeling axioms through  $h_3(\mathbb{O})$  to the chiral SM in a single chain with honest gap identification. Table 1 presents the 9-link chain from self-modeling to chiral fermions.

## 1.1 The derivation chain

Table 1 presents the complete logical chain from self-modeling axioms to the chiral Standard Model representation. The chain consists of nine links, each classified by source and status. One link is a genuine conditioning input (Gap A: Level IV realism + non-composability). Two links are instances of *necessary symmetry reduction* (B1 and B2) in which the reduction is algebraically required and the direction is physically irrelevant (all choices are conjugate). The remaining six links are *proved* within this work and its companions, including the complexification step (Theorem 7).

The result is conditional on one input beyond the self-modeling axioms: the identification of  $h_3(\mathbb{O})$  as the universe algebra via non-composability (Gap A). Given this, the observer’s existence forces the Peirce decomposition (B1:  $F_4 \rightarrow \text{Spin}(9)$ , all idempotents  $F_4$ -conjugate), the observer’s C\*-nature produces complexification (Theorem 7), and the impossibility of equivariant complexification (Schur’s lemma) requires a choice of  $u \in S^6$  (B2:  $G_2 \rightarrow \text{SU}(3)_C$ , all  $u$   $G_2$ -conjugate). Both symmetry reductions are algebraically necessary; both directions are gauge choices.

## 1.2 Paper overview

The paper is organized as follows.

Section 2 (Part A) proves that the C\*-algebra nature of the observer produces complexification of the Peirce half-space  $V_{1/2}$ , upgrading  $\text{Spin}(9) \rightarrow \text{Spin}(10)$  and  $F_4 \rightarrow E_6$ . This covers links L4–L5 of the chain. The complexification is derived from the observer’s C\*-algebra functional calculus (Theorem 7).

Section 3 (Part B) constructs the  $\text{Cl}(6)$  subalgebra from the six directions  $W = u^\perp \cap \text{Im}(\mathbb{O})$  defined by the complex structure  $u$ . The volume form  $\omega_6 = \gamma_1 \cdots \gamma_6$  selects the chiral (left-handed) embedding of the Standard Model fermions via the Pati-Salam [13] intermediate group. This covers link L7.

Section 4 presents the synthesis: the single choice of  $u$  simultaneously determines both the gauge group (via the  $F_4$  intersection route of Todorov–Drenska [8]) and the chiral representation (via  $\text{Cl}(6)$ ). The three factors  $\text{SU}(3)_C$ ,  $\text{SU}(2)_L$ , and  $\text{U}(1)_Y$  are explicitly matched across both routes. This covers links L8–L9.

Section 5 presents a detailed gap analysis. We identify one genuine conditioning input (Gap A), two instances of necessary symmetry reduction (B1, B2), and discuss the open questions of generation structure and the spectral action.

Section 6 places the result in the context of the self-modeling research program (Paper 5 [1]) and discusses connections to the spectral triple approach [2], Dubois-Violette’s earlier work [14], and the broader literature on octonions and exceptional structures in physics [7, 12].

## 2 Part A: Complexification from C\*-Observer

The first half of the derivation proves that the observer’s algebraic nature—specifically, the fact that self-modeling forces a C\*-algebra structure—produces complexification of the Peirce half-space of the exceptional Jordan algebra, upgrading the symmetry from  $\text{Spin}(9)$  to  $\text{Spin}(10)$  and identifying the 16-dimensional spinor representation that will carry one generation of fermions.

### 2.1 The exceptional Jordan algebra $h_3(\mathbb{O})$

The exceptional Jordan algebra  $h_3(\mathbb{O})$  (the Albert algebra) consists of  $3 \times 3$  Hermitian matrices over the octonions  $\mathbb{O}$ :

$$X = \begin{pmatrix} \alpha & \bar{x}_3 & x_2 \\ x_3 & \beta & \bar{x}_1 \\ \bar{x}_2 & x_1 & \gamma \end{pmatrix}, \quad \alpha, \beta, \gamma \in \mathbb{R}, \quad x_1, x_2, x_3 \in \mathbb{O}, \quad (1)$$

with the Jordan product

$$A \circ B = \frac{1}{2}(AB + BA), \quad (2)$$

where  $AB$  denotes ordinary matrix multiplication with octonion entries. The product is well-defined even though  $\mathbb{O}$  is non-associative, by the classical result of Albert [6]; the exceptional Jordan algebra was classified by Jordan, von Neumann, and Wigner [4].

*Remark 1* (Dimension). The three real diagonal entries contribute 3; the three independent octonion off-diagonal entries contribute  $3 \times 8 = 24$ . Hence  $\dim(h_3(\mathbb{O})) = 3 + 24 = 27$ .

*Remark 2* (Non-composability and Gap A). Among the formally real Jordan algebras classified by Jordan–von Neumann–Wigner,  $h_3(\mathbb{O})$  is the unique *exceptional* algebra: it does not embed into any associative algebra and cannot be realized as a tensor factor of a larger system [7]. We identify it as the “universe algebra” on the following grounds. Under Tegmark’s Level IV mathematical universe hypothesis [15] (the program’s foundational premise), if an algebra *can* form composites, then composites of it *exist* as consistent mathematical structures. A system whose algebra admits composites is therefore a subsystem of a larger structure, contradicting its identification as the universe. By contrapositive, the universe’s algebra must be one that does *not* admit composites: non-composability is required, not merely permitted. This argument has an implicit step: identifying “the universe” with a particular mathematical structure, and treating the existence of composites (under L4) as entailing that the algebra is a subsystem *of those composites* rather than merely co-existing alongside them. This identification is natural under L4 (a tensor factor is, by construction, a subsystem of the tensor product) but it is doing substantive philosophical work. A further implicit step is that the universe corresponds to a *single* Jordan algebra rather than, say, a category of algebraic structures; under L4, the non-composable algebra is singled out precisely because every composable algebra is a proper part of a larger structure, but this assumes the universe has a single algebraic description. The argument is flagged as Gap A precisely because it goes beyond the self-modeling axioms.

## 2.2 Peirce decomposition under $E_{11}$

The observer selects a rank-1 idempotent

$$e = E_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3)$$

*Remark 3* (Gap B, step 1). The self-modeling framework requires that the universe contains a faithful self-model—a subsystem whose states separate the states of the whole. Paper 5 proves that any such subsystem has  $M_n(\mathbb{C})^{sa}$  state spaces, i.e. it is a  $C^*$ -algebra observer. An observer of this type, embedded in  $h_3(\mathbb{O})$ , occupies an idempotent whose Peirce 1-space  $V_1$  is its address within the algebra (see Remark 4). The idempotent must be rank-1: a rank-2 idempotent would give  $V_1 \cong h_2(\mathbb{O})$  (10-dimensional), but the observer is a  $C^*$ -*subalgebra* whose address is a single slot, not a composite Jordan system. Only rank-1 gives  $V_1 \cong \mathbb{R}$ , the minimal address for a single observer. All rank-1 idempotents in  $h_3(\mathbb{O})$  are conjugate under  $F_4$  (the orbit is the octonionic projective plane  $\mathbb{O}P^2$ ,  $\dim = 16$ ). The observer’s existence breaks  $F_4 \rightarrow \text{Spin}(9)$ ; *which* idempotent is selected corresponds to the observer’s “viewpoint” and is physically irrelevant by  $F_4$ -conjugacy. The breaking is forced by the self-modeling definition (which requires an observer); the direction is gauge.

*Remark 4* (Observer-universe relation). Paper 5 shows that a self-modeling system’s state space is  $M_n(\mathbb{C})^{sa}$ , which excludes  $h_3(\mathbb{O})$  as an observer algebra. This is *not* a contradiction:

the observer is not the universe. The observer is *internal* to  $h_3(\mathbb{O})$  via the Peirce decomposition:  $V_1 \cong \mathbb{R}$  is the observer's slot,  $V_{1/2}$  is the interaction space, and  $V_0$  is the complement. The Peirce decomposition is the Jordan-algebraic analog of a subsystem decomposition—not a tensor-product factorization (which  $h_3(\mathbb{O})$  does not admit, by non-composability) but a spectral decomposition under the observer's idempotent. The observer's *internal* algebra (its state space as a self-modeling system) is  $C^*$  by Paper 5; the *universe's* algebra is the non-composable  $h_3(\mathbb{O})$ . The observer accesses the universe through the Peirce interface  $V_{1/2}$ , not through tensor factors.

The dimensional relationship deserves emphasis. The Peirce 1-space  $V_1 = \mathbb{R} \cdot E_{11}$  is the observer's *address* within  $h_3(\mathbb{O})$ —the eigenvalue-1 slot of the idempotent it occupies—not the observer's internal state space. The observer's internal  $C^*$ -algebra must be rich enough to faithfully represent the Peirce operators  $T_a \in \text{End}(V_{1/2})$ , which generate  $M_{16}(\mathbb{R})$ . Paper 5 forces this internal algebra to be  $M_n(\mathbb{C})^{\text{sa}}$  for  $n \geq 16$ ; the observer's *model* of the Peirce interface lives in  $M_{16}(\mathbb{C}) \subset M_n(\mathbb{C})$ . The 1-dimensional Peirce slot and the  $16^2$ -dimensional measurement algebra are not in conflict: the former is where the observer *sits* in  $h_3(\mathbb{O})$ , the latter is how it *represents* the interface  $V_{1/2}$  internally. This is the standard situation for a subsystem probing its environment: the subsystem's address is low-dimensional, but its internal model of the environment is as rich as the information accessible through the interface requires.

The Peirce decomposition under  $e$  partitions  $h_3(\mathbb{O})$  into the eigenspaces of the multiplication operator  $L_e(X) = e \circ X$ :

$$h_3(\mathbb{O}) = V_1 \oplus V_{1/2} \oplus V_0, \quad V_\lambda = \{X \in h_3(\mathbb{O}) : e \circ X = \lambda X\}. \quad (4)$$

**Explicit computation of  $L_e(X)$ .** Writing  $E_{11} X + X E_{11}$  entry by entry and dividing by 2:

$$e \circ X = \frac{1}{2} \begin{pmatrix} 2\alpha & \bar{x}_3 & x_2 \\ x_3 & 0 & 0 \\ \bar{x}_2 & 0 & 0 \end{pmatrix}. \quad (5)$$

Reading off the eigenvalue conditions:

**Proposition 5** (Peirce spaces). *The Peirce decomposition of  $h_3(\mathbb{O})$  under  $E_{11}$  is:*

$$V_1 = \{\alpha E_{11} : \alpha \in \mathbb{R}\} \cong \mathbb{R}, \quad \dim(V_1) = 1, \quad (6)$$

$$V_{1/2} = \left\{ \begin{pmatrix} 0 & \bar{x}_3 & x_2 \\ x_3 & 0 & 0 \\ \bar{x}_2 & 0 & 0 \end{pmatrix} : x_2, x_3 \in \mathbb{O} \right\} \cong \mathbb{O}^2, \quad \dim(V_{1/2}) = 16, \quad (7)$$

$$V_0 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & \bar{x}_1 \\ 0 & x_1 & \gamma \end{pmatrix} : \beta, \gamma \in \mathbb{R}, x_1 \in \mathbb{O} \right\} \cong h_2(\mathbb{O}), \quad \dim(V_0) = 10. \quad (8)$$

**Dimension check:**  $1 + 16 + 10 = 27 = \dim(h_3(\mathbb{O}))$ .  $\checkmark$

**Stabilizer and Spin(9) representations.** The stabilizer of  $E_{11}$  in  $F_4 = \text{Aut}(h_3(\mathbb{O}))$  ( $\dim F_4 = 52$ ) is [7, 12]

$$\text{Stab}_{F_4}(E_{11}) = \text{Spin}(9), \quad \dim \text{Spin}(9) = 36 = \binom{9}{2}. \quad (9)$$

Coset check:  $52 - 36 = 16 = \dim(V_{1/2})$ , consistent with  $F_4/\text{Spin}(9) \cong \mathbb{O}P^2$ .

Under  $\text{Spin}(9)$  the Peirce spaces carry the representations:

Peirce space	Identification	$\dim_{\mathbb{R}}$	$\text{Spin}(9)$ rep
$V_1$	$\mathbb{R} \cdot E_{11}$	1	trivial $\mathbf{1}$
$V_{1/2}$	$\mathbb{O}^2$	16	spinor $S_9 = \mathbf{16}$
$V_0$	$h_2(\mathbb{O})$	10	$\mathbf{9} \oplus \mathbf{1}$
Total	$h_3(\mathbb{O})$	27	$\mathbf{1} \oplus \mathbf{16} \oplus \mathbf{9} \oplus \mathbf{1}$

The identification  $V_{1/2} = \mathbb{O}^2 = S_9$  (the unique real 16-dimensional irreducible spinor of  $\text{Spin}(9)$ ) follows from  $\dim_{\mathbb{R}}(S_9) = 2^{\lfloor 9/2 \rfloor} = 2^4 = 16$  and the fact that  $\text{Spin}(9)$  acts on the tangent space  $T_{E_{11}}(\mathbb{O}P^2) \cong \mathbb{O}^2$  via its spinor representation [7, 12]. The space  $V_0 = h_2(\mathbb{O})$  decomposes into the traceless part (the 9-dimensional vector representation of  $\text{SO}(9) = \text{Spin}(9)/\mathbb{Z}_2$ ) plus the trace (a trivial  $\mathbf{1}$ ).

**Peirce multiplication operators.** For each unit vector  $a$  in the 9-dimensional vector representation (the traceless part of  $V_0 = h_2(\mathbb{O})$ ), the *Peirce multiplication operator* is

$$T_a: V_{1/2} \rightarrow V_{1/2}, \quad T_a(v) = (a \circ v)_{V_{1/2}}, \quad (10)$$

where the subscript  $V_{1/2}$  denotes projection to the Peirce half-space. These are self-adjoint with respect to the trace inner product on  $h_3(\mathbb{O})$ . For an orthonormal basis  $\{a_1, \dots, a_9\}$  of the vector representation,

$$\{T_{a_i}, T_{a_j}\} = \frac{1}{2} \delta_{ij} I_{16}, \quad (11)$$

so  $T_a^2 = \frac{1}{4} I_{16}$  and the rescaled operators  $\gamma_i = 2T_{a_i}$  satisfy the standard Clifford relation  $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$ . The image of the Clifford algebra  $\text{Cl}(9, 0)$  in  $\text{End}(V_{1/2}) = M_{16}(\mathbb{R})$  is  $M_{16}(\mathbb{R})$  itself: the abstract algebra  $\text{Cl}(9, 0) \cong M_{16}(\mathbb{R}) \oplus M_{16}(\mathbb{R})$  has two simple summands, and the spinor representation  $S_9$  is the irreducible module for one summand.

## 2.3 Complexification from C\*-observer nature

The central claim of Part A is that *complexification is a consequence of the observer's algebraic nature, not assumed*. We prove this via the continuous functional calculus of the observer's C\*-algebra, applied to the Peirce operators.

**Step 1: Self-modeling forces a C\*-algebra observer.** Paper 5 [1] defines a *self-modeling system* as a finite-dimensional order-unit space  $(V, e)$  containing a sub-order-unit space  $(\hat{V}, \hat{e})$  satisfying four axioms: (A1) the states of  $\hat{V}$  separate the states of  $V$  (faithfulness); (A2)  $\hat{V}$  is updated through the boundary between the subsystem and its complement; (A3) the sequential product  $a \& b = \sqrt{a} b \sqrt{a}$  defines the measurement composition; (A4)  $\hat{V}$  is closed under this product. Paper 5 proves that these axioms force  $V \cong M_n(\mathbb{C})^{\text{sa}}$  for some  $n$ —the self-adjoint part of a matrix C\*-algebra.

The square root in the sequential product is computed via the *continuous functional calculus* (CFC) of the ambient C\*-algebra  $M_n(\mathbb{C})$ . For a self-adjoint element  $x$  with spectral

decomposition  $x = \sum_i \lambda_i P_i$ , this calculus defines  $f(x) = \sum_i f(\lambda_i) P_i$  for any continuous  $f: \sigma(x) \rightarrow \mathbb{C}$ . This is the *scalar* functional calculus: it applies  $f$  to each eigenvalue independently and is uniquely determined by the  $C^*$ -algebra structure. The distinction between this scalar calculus and general matrix-function constructions is critical for Section 2.3 (see Remark 8).

**Step 2: Peirce operators have negative eigenvalues.** The Peirce multiplication operators  $T_a$  (Section 2.2) satisfy  $T_a^2 = \frac{1}{4}I_{16}$ , so each  $T_a$  has spectrum  $\{+\frac{1}{2}, -\frac{1}{2}\}$ . The eigenvalue  $-\frac{1}{2}$  is *negative*. In a real algebra,  $\sqrt{-\frac{1}{2}}$  is undefined. But the observer is  $M_n(\mathbb{C})^{\text{sa}}$  (Step 1), so the continuous functional calculus of  $M_n(\mathbb{C})$  applies the principal branch:

$$\sqrt{-\frac{1}{2}} = \frac{i}{\sqrt{2}}. \quad (12)$$

This is the principal branch value ( $\arg(\sqrt{z}) \in (-\pi/2, \pi/2]$ ); the sign of  $i$  depends on branch convention (see below), but the exit from  $\mathbb{R}$  does not.

**Step 3: The CFC computation exits  $M_{16}(\mathbb{R})$ .** The observer faithfully represents the Peirce operators as self-adjoint elements of  $M_n(\mathbb{C})$  via the standard inclusion  $\iota: M_{16}(\mathbb{R}) \hookrightarrow M_{16}(\mathbb{C})$  ( $n \geq 16$ ). The spectral decomposition of  $T_a$  gives

$$\sqrt{T_a} = \frac{1}{\sqrt{2}}(P_+ + iP_-), \quad (13)$$

where  $P_{\pm}$  are the spectral projections onto the  $\pm\frac{1}{2}$  eigenspaces.

**Lemma 6** (CFC well-definedness). *Let  $T_a, T_b$  be self-adjoint Peirce operators, faithfully represented in the observer's  $C^*$ -algebra  $M_n(\mathbb{C})$  via  $\iota: M_{16}(\mathbb{R}) \hookrightarrow M_n(\mathbb{C})$ . Then:*

- (a)  $\sqrt{\iota(T_a)} \iota(T_b) \sqrt{\iota(T_a)}$  is well-defined via the continuous functional calculus of  $M_n(\mathbb{C})$ .
- (b) For effects  $a \in [0, I]$ , this operation coincides with the sequential product  $a \& b = \sqrt{a} b \sqrt{a}$  derived in Paper 5 [1].

*Proof.* (a) For any self-adjoint  $x \in M_n(\mathbb{C})^{\text{sa}}$  with spectral decomposition  $x = \sum_i \lambda_i P_i$ , the continuous functional calculus defines  $f(x) = \sum_i f(\lambda_i) P_i$  for every continuous  $f: \sigma(x) \rightarrow \mathbb{C}$ . Applying this to  $f(z) = z^{1/2}$  on  $\sigma(\iota(T_a)) = \{+\frac{1}{2}, -\frac{1}{2}\}$  yields  $\sqrt{\iota(T_a)} \in M_n(\mathbb{C})$  (the branch choice on  $-\frac{1}{2}$  is addressed below Theorem 7). The triple product  $\sqrt{\iota(T_a)} \iota(T_b) \sqrt{\iota(T_a)}$  is then a well-defined element of  $M_n(\mathbb{C})$ . (b) Paper 5 derives the Lüders form  $a \& b = \sqrt{a} b \sqrt{a}$  for the sequential product on effects, where  $\sqrt{a}$  is the unique positive square root—itself a CFC element. For effects, the positive square root and the CFC square root coincide, so the two computations are identical.  $\square$

**Theorem 7** (Observer-induced complexification). *Let  $T_a, T_b$  be anticommuting Peirce operators ( $\{T_a, T_b\} = 0, a \neq b$ ) in  $\text{End}(V_{1/2}) = M_{16}(\mathbb{R})$ , and let  $\iota: M_{16}(\mathbb{R}) \hookrightarrow M_{16}(\mathbb{C}) \subset M_n(\mathbb{C})$  be the faithful representation of the Peirce operators inside the observer's  $C^*$ -algebra (Step 3). Then the operation  $\sqrt{T_a} T_b \sqrt{T_a}$  (Lemma 6), computed via the continuous functional calculus of  $M_n(\mathbb{C})$ , gives*

$$\sqrt{\iota(T_a)} \iota(T_b) \sqrt{\iota(T_a)} = \frac{i}{2} \iota(T_b). \quad (14)$$

We write  $\sqrt{T_a}$  for  $\sqrt{\iota(T_a)}$  when the representation context is clear.

*Proof.* Expand using (13):

$$\sqrt{T_a} T_b \sqrt{T_a} = \frac{1}{2}(P_+ + iP_-) T_b (P_+ + iP_-) = \frac{1}{2}(P_+ T_b P_+ + iP_+ T_b P_- + iP_- T_b P_+ - P_- T_b P_-). \quad (15)$$

From  $\{T_a, T_b\} = 0$  and  $P_+ T_a = \frac{1}{2} P_+$ , it follows that  $P_+ T_b P_+ = 0$  and  $P_- T_b P_- = 0$  (anticommuting Hermitian matrices have purely off-diagonal structure in each other's eigenbasis). Therefore  $T_b = P_+ T_b P_- + P_- T_b P_+$ , and

$$\sqrt{T_a} T_b \sqrt{T_a} = \frac{i}{2}(P_+ T_b P_- + P_- T_b P_+) = \frac{i}{2} T_b. \quad (16)$$

□

The specific sign of  $i$  in (14) depends on the branch convention (replacing  $i$  by  $-i$  corresponds to choosing the other continuous branch of  $\sqrt{z}$  on  $\{+\frac{1}{2}, -\frac{1}{2}\}$ ), but the exit from  $\mathbb{R}$  to  $\mathbb{C}$  does not: every square-root choice for eigenvalue  $-\frac{1}{2}$  produces a non-real value, so the measurement algebra exits  $M_{16}(\mathbb{R})$  regardless of convention.

The factor of  $i$  is determined by the functional calculus, not conventional. Paper 5 establishes that a self-modeling observer's measurement composition is the sequential product  $a \& b = \sqrt{a} b \sqrt{a}$  for effects, computed via the continuous functional calculus of its C\*-algebra  $M_n(\mathbb{C})$ . The same functional calculus applies to all self-adjoint elements of  $M_n(\mathbb{C})$ , including the Peirce operators  $T_a$  (Lemma 6). When the observer probes the Peirce half-space via the operators  $T_a$ —which are the natural observables of the Peirce interface (Eq. (10))—the CFC computation  $\sqrt{T_a} T_b \sqrt{T_a}$  necessarily involves  $\sqrt{T_a}$ . Since  $T_a$  has eigenvalue  $-\frac{1}{2} < 0$ , the real functional calculus cannot produce  $\sqrt{T_a}$ ; only the complex functional calculus applies (Eq. (12)). By contrast, the effect operators  $E_a = \frac{1}{2}(I_{16} + 2T_a)$  have spectrum  $\{0, 1\} \subset [0, 1]$ , so  $\sqrt{E_a}$  is real-valued and the sequential product  $E_a \& E_b = \frac{1}{2} E_a$  remains entirely real. The complexification arises specifically from the negative eigenvalues of the Peirce operators, not from a choice to extend beyond the effect domain.

*Remark 8* (Scalar functional calculus vs. matrix square roots). A potential objection is that  $T_a$  admits a *real* matrix square root: since the eigenvalue  $-\frac{1}{2}$  has even multiplicity (8), one can pair eigenspaces into  $2 \times 2$  blocks and use real rotation matrices  $J$  ( $J^2 = -I_2$ ) to construct  $S \in M_{16}(\mathbb{R})$  with  $S^2 = T_a$  (cf. [16], Thm. 1.29).

This objection conflates two distinct notions of “square root.” The C\*-algebra continuous functional calculus (Step 1) is the *scalar* functional calculus: it defines  $\sqrt{T_a} = \sum_i \sqrt{\lambda_i} P_i$ , applying the square-root function to each eigenvalue *independently*. For  $\lambda = -\frac{1}{2}$ , the equation  $z^2 = -\frac{1}{2}$  has no real solution, so the functional calculus square root is necessarily complex. The real matrix square root  $S$  described above is *not* a functional calculus element: it cannot be written as  $g(T_a) = \sum_i g(\lambda_i) P_i$  for any scalar function  $g$ , because it mixes eigenspaces via  $2 \times 2$  rotation blocks rather than acting on each eigenspace by a scalar.

Paper 5 establishes that the self-modeling sequential product is computed via the C\*-algebra continuous functional calculus. This is not a choice among available square roots but a consequence of the C\*-algebraic structure: the functional calculus is uniquely determined, and it is the scalar one. The Higham-type real matrix square roots, while mathematically valid, are not the square roots that the self-modeling framework selects.

*Remark 9* (Why the observer measures  $T_a$ , not just effects). A natural objection is that the Gudder–Greechie sequential product [17] is defined for effects (spectrum in  $[0, 1]$ ), while  $T_a$

has eigenvalue  $-\frac{1}{2}$ . Lemma 6 establishes that  $\sqrt{T_a} T_b \sqrt{T_a}$  is well-defined in the observer's  $C^*$ -algebra via the continuous functional calculus, and that this operation coincides with the sequential product on effects. Here we address why this computation is physically relevant for the Peirce operators, which lie outside the effect interval.

*$T_a$  is the algebraic primitive;  $E_a$  is a derived rescaling.* The Peirce operator  $T_a(v) = (a \circ v)_{V_{1/2}}$  is the Jordan product restricted to  $V_{1/2}$ —the fundamental algebraic operation at the observer's interface with the universe. The effect  $E_a = \frac{1}{2}(I + 2T_a)$  is an affine transformation of  $T_a$  into the unit interval—a derived object, not the fundamental one. The objection that “the observer should use  $E_a$  instead of  $T_a$ ” reverses the logical order.

*$T_a$  and  $E_a$  are different observables.*  $T_a$  has eigenvalues  $\pm\frac{1}{2}$ ;  $E_a$  has eigenvalues  $\{0, 1\}$ . They represent different measurement outcomes, and the CFC computations  $\sqrt{T_a} T_b \sqrt{T_a}$  and  $\sqrt{E_a} E_b \sqrt{E_a}$  encode different physical compositions. Shifting the spectrum is not a gauge freedom—it changes what is being measured.

*$T_a$  is in the algebra necessarily.* The Peirce operators generate  $\text{End}(V_{1/2}) = M_{16}(\mathbb{R})$ : they are the basis of the algebra of observable operations on the Peirce interface. Any faithful representation of this algebra in the observer's  $C^*$ -algebra  $M_n(\mathbb{C})$  must include the  $T_a$  as self-adjoint elements. Once  $T_a \in M_n(\mathbb{C})^{\text{sa}}$ , the continuous functional calculus defines  $\sqrt{T_a}$  uniquely, and the CFC computation  $\sqrt{T_a} T_b \sqrt{T_a}$  is a well-defined element of  $M_n(\mathbb{C})$ . The complexification is therefore a structural consequence of the algebra's  $C^*$ -nature—it follows from the *existence* of  $T_a$  in the algebra, not from any particular measurement the observer performs.

*Remark 10* (Sequential product output and the generated algebra). Theorem 7 produces  $\sqrt{T_a} T_b \sqrt{T_a} = (i/2) T_b$ , which is anti-self-adjoint—not an observable in the standard sense. This is not a defect of the argument but its operational content.

The CFC operation  $\sqrt{T_a} (\cdot) \sqrt{T_a}$  acts within the observer's  $C^*$ -algebra  $M_n(\mathbb{C})$ ; its closure generates a *subalgebra*, not merely a collection of self-adjoint elements. The subalgebra generated by  $\{T_a\}$  together with all iterated CFC products is  $M_{16}(\mathbb{C})$ , not  $M_{16}(\mathbb{R})$ . The physical consequence is that the observer's *observable space*—the self-adjoint part of the generated algebra—is  $M_{16}(\mathbb{C})^{\text{sa}}$ , which is strictly larger than  $M_{16}(\mathbb{R})$ . Elements such as  $T_a + (i/2) T_b$  are self-adjoint in  $M_{16}(\mathbb{C})$  but have no counterpart in  $M_{16}(\mathbb{R})$ : they are new observables that exist only because the algebra extends to  $M_{16}(\mathbb{C})$ .

The complexification is therefore a statement about the *algebra of all composable measurements*, not about any single measurement outcome. The symmetries of the Peirce interface are those of this generated algebra— $\text{Spin}(10)$  acting on  $\mathbb{C}^{16}$ —not those of the original real representation.

The algebra generated by the observer's CFC computations  $\sqrt{T_a} T_b \sqrt{T_a}$  on the Peirce operators is therefore  $M_{16}(\mathbb{C})$ , not  $M_{16}(\mathbb{R})$ : the  $\mathbb{C}$ -linear span of all monomials in the  $T_a$  and their CFC products has complex dimension 256, equal to  $\dim_{\mathbb{C}} M_{16}(\mathbb{C})$ . This is the image of the complexified Clifford algebra  $\text{Cl}(9, \mathbb{C}) \cong M_{16}(\mathbb{C}) \oplus M_{16}(\mathbb{C})$  acting on  $V_{1/2}^{\mathbb{C}} = \mathbb{C}^{16}$  via one of its two simple summands (cf. the real case in Eq. (11)). The observer's effective description of  $V_{1/2}$  is therefore

$$V_{1/2}^{\mathbb{C}} := V_{1/2} \otimes_{\mathbb{R}} \mathbb{C}, \quad \dim_{\mathbb{C}}(V_{1/2}^{\mathbb{C}}) = 16. \quad (17)$$

*Remark 11* (Compatibility with impossibility theorems). No  $\text{Spin}(9)$ -equivariant complex structure  $J : V_{1/2} \rightarrow V_{1/2}$  with  $J^2 = -\text{Id}$  exists (Schur's lemma:  $\text{End}_{\text{Spin}(9)}(S_9) = \mathbb{R}$ , since

$9 \bmod 8 = 1$  gives real type by Bott periodicity). The observer-induced complexification is *non-equivariant*: it depends on which  $T_a$  the observer measures first. Different choices of  $T_a$  produce different complex structures, corresponding to different choices of  $u \in S^6 \subset \text{Im}(\mathbb{O})$  (Gap B, step 2). The correspondence is explicit: each Peirce operator  $T_a$  is indexed by a unit vector  $a$  in the 9-dimensional traceless part of  $V_0 = h_2(\mathbb{O})$ ; for  $a = e_k \in \text{Im}(\mathbb{O})$  ( $k = 1, \dots, 7$ ), the complex structure induced on  $V_{1/2} = \mathbb{O}^2$  by the CFC computation is left multiplication by  $e_k$ , giving  $u = e_k$ . All seven such complex structures are  $G_2$ -conjugate (orbit =  $S^6$ , dim 6). The impossibility of equivariant complexification is complemented, not circumvented, by the observer’s external input.

*Remark 12* (Complexification vs. complex structure). Theorem 7 establishes that complexification *occurs* for any pair of anticommuting Peirce operators—no choice of  $u$  is required. The choice of  $u \in S^6$  (Gap B, step 2) determines *which* complex structure the observer obtains, not *whether* complexification occurs. These are logically distinct: the former (Gap C) is closed unconditionally by Theorem 7; the latter (Gap B, step 2) is a necessary symmetry reduction forced by the combination of Theorem 7 and Schur’s lemma (Remark 11). Since all  $u \in S^6$  are  $G_2$ -conjugate, the reduction is required but the direction is physically irrelevant—a gauge choice (cf. [18] for an explicit  $G_2 \rightarrow \text{SU}(3)$  model on  $S^6$ ).

Crucially, B2 does not await external dynamics. In the self-modeling framework, the observer’s *only* dynamics is the sequential product: the “before  $\rightarrow$  model  $\rightarrow$  after” cycle of self-modeling measurement (Paper 5). Moreover, the observer *must* probe  $V_{1/2}$ : Paper 5 requires a faithful self-model updated through the system’s boundary, and  $V_{1/2}$  *is* the Peirce boundary—the only sector through which the observer’s slot  $V_1$  communicates with the complement  $V_0$  (the Peirce multiplication rules give  $V_1 \circ V_0 = 0$ , so  $V_{1/2}$  is the sole channel). An observer that ignores  $V_{1/2}$  has no model of its environment and fails the faithfulness condition.

Given that  $V_{1/2}$  must be probed, the observer cannot do so without applying  $\sqrt{T_a}(\cdot)\sqrt{T_a}$  for some specific  $T_a$ ; *every* such application involves a specific  $T_a$  and thereby a specific  $u$ . There is no state of the self-modeling cycle in which  $V_{1/2}$  is accessed but no  $T_a$  has been selected: probing  $V_{1/2}$  and selecting a  $T_a$  are the same algebraic act. That the observer must access  $V_{1/2}$  via *some*  $T_a$  is forced; *which*  $T_a$ —and hence which  $u \in S^6$ —is not determined by the algebra, which is what makes B2 a genuine necessary symmetry reduction rather than a derived result. The observer’s C\*-nature—expressed through the CFC computation  $\sqrt{T_a}(\cdot)\sqrt{T_a}$ —is therefore both the mechanism that produces complexification (Theorem 7) and the act that selects the complex structure. No temporal ordering is required—the point is structural: any instance of the self-modeling cycle that accesses  $V_{1/2}$  necessarily involves a specific  $T_a$ , hence a specific  $u$ .

*Remark 13* (Contrast with Boyle [10]). Boyle starts from the complexified algebra  $h_3^{\mathbb{C}}(\mathbb{O}) := h_3(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$  and studies its structure group  $E_6$ . Our approach differs in that we *derive* the complexification from the observer’s C\*-nature: the self-modeling requirement (Paper 5) forces the observer to be a C\*-algebra system, and the continuous functional calculus applied to the Peirce operators necessarily exits  $M_{16}(\mathbb{R})$  into  $M_{16}(\mathbb{C})$  (Theorem 7). The end result is the same identification of  $V_{1/2}^{\mathbb{C}}$  as the Weyl spinor  $S_{10}^+$ , but the logical route is different: Boyle assumes complexification; we derive it from self-modeling.

## 2.4 Spin(9) to Spin(10) and $F_4$ to $E_6$

The complexification of  $V_{1/2}$  triggers a representation-theoretic upgrade from Spin(9) to Spin(10).

**Proposition 14** (Spinor upgrade). *The complexified Peirce half-space  $V_{1/2}^{\mathbb{C}} = \mathbb{O}^2 \otimes_{\mathbb{R}} \mathbb{C}$  carries the Weyl spinor representation  $S_{10}^+$  of Spin(10):*

$$S_{10}^+|_{\text{Spin}(9)} \cong S_9 \otimes_{\mathbb{R}} \mathbb{C} = S_9^{\mathbb{C}}. \quad (18)$$

*Proof.* Spin(10) has two inequivalent complex Weyl representations  $S_{10}^+$  and  $S_{10}^-$ , each of  $\dim_{\mathbb{C}} = 2^{\lfloor 10/2 \rfloor - 1} = 2^4 = 16$ . The standard branching rule for the embedding Spin(9)  $\hookrightarrow$  Spin(10) (where Spin(9) stabilizes a unit vector in  $\mathbb{R}^{10}$ ) gives  $S_{10}^+|_{\text{Spin}(9)} \cong S_9^{\mathbb{C}}$ : the Weyl spinor of Spin(10), restricted to Spin(9), is the complexification of the real spinor  $S_9$ . Equivalently,  $S_9$  is a real form of  $S_{10}^+$ .

Since  $V_{1/2} = S_9$  (Proposition 5) and  $V_{1/2}^{\mathbb{C}} = S_9^{\mathbb{C}}$ , we conclude  $V_{1/2}^{\mathbb{C}} = S_{10}^+$ .  $\square$

**Dimension check:**  $\dim_{\mathbb{C}}(S_{10}^+) = 16 = \dim_{\mathbb{R}}(S_9) = \dim_{\mathbb{R}}(V_{1/2})$ .  $\checkmark$

*Remark 15* (Why Spin(10), not just complex Spin(9)). A natural objection is that complexifying  $S_9$  gives a complex Spin(9)-module, not automatically a Spin(10)-module. The upgrade is structural, not merely representation-theoretic.

*Representation-theoretic uniqueness.* The branching  $S_{10}^+|_{\text{Spin}(9)} \cong S_9^{\mathbb{C}}$  is multiplicity-free:  $S_9^{\mathbb{C}}$  is irreducible under Spin(9). By Schur's lemma, any Spin(10)-module structure on  $S_9^{\mathbb{C}}$  that restricts to the given Spin(9) action is unique (up to the  $S_{10}^+$  vs.  $S_{10}^-$  choice resolved below).

*Structural forcing.* The decisive point is not representation compatibility but the structure group of  $h_3^{\mathbb{C}}(\mathbb{O})$ . The complexification  $h_3(\mathbb{O}) \rightarrow h_3^{\mathbb{C}}(\mathbb{O})$  upgrades the automorphism group from  $F_4$  to the structure group  $E_6$ , and correspondingly upgrades the stabilizer of  $E_{11}$  from Spin(9) to Spin(10)  $\times$  U(1) (Eq. (21)). The additional 9 generators of  $\mathfrak{spin}(10)$  beyond  $\mathfrak{spin}(9)$  are structure-group elements of  $h_3^{\mathbb{C}}(\mathbb{O})$  that do not preserve the real  $h_3(\mathbb{O})$ : they exist in  $M_{16}(\mathbb{C})$  but not in  $M_{16}(\mathbb{R})$ , and they are *constructed* by the complexification that Theorem 7 establishes. Thus Spin(10) is the symmetry of the observer's complexified description of the Peirce interface—the stabilizer of  $E_{11}$  within the structure group of the algebra the observer actually works in—not merely a group whose representation ring happens to contain  $S_9^{\mathbb{C}}$ .

*Why the upgrade is physical, not merely formal.* The observer cannot “ignore” the complexification and retain Spin(9) as the relevant symmetry group. Theorem 7 establishes that the observer's measurement algebra is  $M_{16}(\mathbb{C})$ , not  $M_{16}(\mathbb{R})$ : the CFC computations it performs generate complex elements that do not lie in  $M_{16}(\mathbb{R})$ . The symmetry group of the observer's description of  $V_{1/2}$  is therefore the stabilizer of  $E_{11}$  in the structure group of  $h_3^{\mathbb{C}}(\mathbb{O})$  (which is  $E_6$ ), not in the automorphism group of  $h_3(\mathbb{O})$  (which is  $F_4$ ). This gives Spin(10)  $\times$  U(1) (Eq. (21)), not Spin(9) (Eq. (9)).

*Remark 16* (Choice of  $S_{10}^+$  vs.  $S_{10}^-$ ). Under Spin(10)  $\supset$  Spin(9), both Weyl representations restrict to the same real representation  $S_9$  (since  $S_9$  has no chirality). The assignment  $S_{10}^+$  (rather than  $S_{10}^-$ ) follows the Boyle convention [10]. The chirality distinction will be resolved in Section 3 via the Cl(6) construction.

**Symmetry upgrade.** Because  $V_{1/2}^{\mathbb{C}} = S_{10}^+$  carries the unique  $\text{Spin}(10)$ -module structure extending the  $\text{Spin}(9)$  action (Remark 15), the symmetry of the observer's description of the Peirce space upgrades:

$$\text{Spin}(9) \longrightarrow \text{Spin}(10). \quad (19)$$

At the algebra level, the complexification extends from  $V_{1/2}$  to the full algebra  $h_3(\mathbb{O})$ . The Peirce multiplication rules  $V_{1/2} \circ V_{1/2} \subset V_1 \oplus V_0$  mean that Jordan products of elements in  $V_{1/2}^{\mathbb{C}}$  land in  $V_1^{\mathbb{C}} \oplus V_0^{\mathbb{C}}$ : the observer's complex description of the interface forces a complex description of every sector reached by interface products. Concretely, Theorem 7 produces genuinely complex elements  $v = x + iy \in V_{1/2}^{\mathbb{C}}$  with  $y \neq 0$ ; their Jordan products  $v \circ v$  have imaginary part  $2(x \circ y)_{V_1 \oplus V_0}$ , which is generically nonzero (it vanishes only on a measure-zero set of pairs), so the complex extension of  $V_0$  and  $V_1$  is used nontrivially, not merely formally. Since the Peirce multiplications generate all of  $h_3(\mathbb{O})$ , the observer's effective algebra is  $h_3^{\mathbb{C}}(\mathbb{O}) = h_3(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$ , not merely  $V_{1/2}^{\mathbb{C}}$ . The structure group of the complexified algebra is [7, 10, 12]

$$F_4 \longrightarrow E_6, \quad \dim F_4 = 52 \rightarrow \dim E_6 = 78. \quad (20)$$

These are different kinds of symmetry groups:  $F_4 = \text{Aut}(h_3(\mathbb{O}))$  is the automorphism group (preserving both the Jordan product and the unit element), while  $E_6 = \text{Str}_0(h_3^{\mathbb{C}}(\mathbb{O}))$  is the structure group of the complexified algebra (preserving the cubic determinant form up to scale). The stabilizer of  $E_{11}$  in  $E_6$  is

$$\text{Stab}_{E_6}(E_{11}) = \text{Spin}(10) \times \text{U}(1), \quad \dim = 45 + 1 = 46. \quad (21)$$

Under  $\text{Spin}(10)$ , the 27-dimensional representation decomposes as

$$\mathbf{27} \rightarrow \underbrace{\mathbf{1}}_{V_1^{\mathbb{C}}} \oplus \underbrace{\mathbf{10}}_{V_0^{\mathbb{C}}} \oplus \underbrace{\mathbf{16}}_{V_{1/2}^{\mathbb{C}}=S_{10}^+}, \quad (22)$$

where  $V_1^{\mathbb{C}} \cong \mathbb{C}$  (trivial),  $V_0^{\mathbb{C}} \cong \mathbf{10}$  (vector of  $\text{Spin}(10)$ , verified by the branching  $\mathbf{10}|_{\text{Spin}(9)} = \mathbf{9} \oplus \mathbf{1}$ ), and  $V_{1/2}^{\mathbb{C}} = S_{10}^+$  (Weyl spinor).

**Dimension check:**  $1 + 10 + 16 = 27$ . ✓

*Remark 17* (Cross-check with Boyle). Boyle [10] arrives at the same decomposition  $\mathbf{27} = \mathbf{1} \oplus \mathbf{10} \oplus \mathbf{16}$  under  $\text{Spin}(10) \subset E_6$  by analyzing the complexified algebra directly. The results are consistent; our contribution is to trace the complexification step to the *combination* of  $\text{C}^*$ -observer nature and the negative-spectrum Peirce operators (Theorem 7), rather than taking it as a mathematical starting point.

*Remark 18* (Scope of the complexification proof). Theorem 7 establishes a precise algebraic fact: the CFC computation  $\sqrt{T_a} T_b \sqrt{T_a}$  on anticommuting Peirce operators produces imaginary coefficients, extending the generated algebra from  $M_{16}(\mathbb{R})$  to  $M_{16}(\mathbb{C})$ . This is not a physical heuristic but a computation within the  $\text{C}^*$ -algebra that the observer inhabits (Paper 5). The key input is that the Peirce operators  $T_a$  have eigenvalue  $-\frac{1}{2} < 0$ : in a real algebra,  $\sqrt{-\frac{1}{2}}$  is undefined; in the observer's complex  $\text{C}^*$ -algebra, it is  $i/\sqrt{2}$ . The exit from  $\mathbb{R}$  to  $\mathbb{C}$  is produced by the *combination* of the observer's  $\text{C}^*$ -nature (Paper 5) and the spectral

properties of the Peirce operators (the  $h_3(\mathbb{O})$  structure). Neither input alone suffices: a real-algebraic observer would have no complex functional calculus; an operator with non-negative spectrum would have a real square root.

The upgrade from  $\text{Spin}(9)$  to  $\text{Spin}(10)$  then follows from the multiplicity-free branching  $S_{10}^+|_{\text{Spin}(9)} \cong S_9^{\mathbb{C}}$  (Remark 15): the complexified space  $S_9^{\mathbb{C}}$  admits a unique  $\text{Spin}(10)$ -module structure extending the  $\text{Spin}(9)$  action, whose existence is guaranteed because  $S_9$  is the restriction of  $S_{10}^+$  to  $\text{Spin}(9) = \text{Stab}_{\text{Spin}(10)}(e_{10})$ .

*Interpretive independence.* The complexification result requires exactly two mathematical inputs: (i) the Peirce operators  $T_a$  are faithfully represented as self-adjoint elements of a  $C^*$ -algebra  $M_n(\mathbb{C})$ , and (ii) the square root is computed via the  $C^*$ -algebra continuous functional calculus. Input (i) follows from any faithful embedding of  $M_{16}(\mathbb{R})$  into a complex matrix algebra. Input (ii) is a standard  $C^*$ -algebra property. The physical interpretation of the sequential product (as measurement composition, per Paper 5) motivates why  $\sqrt{T_a} T_b \sqrt{T_a}$  is the relevant operation, but the algebraic fact that it produces imaginary coefficients depends only on (i) and (ii).

### 3 Part B: $\text{Cl}(6)$ Chirality and the Standard Model

The second half of the derivation shows that the same algebraic structure—the exceptional Jordan algebra probed by a  $C^*$ -observer—yields the Standard Model gauge group with the correct chiral representation. The key mechanism is a Clifford subalgebra  $\text{Cl}(6)$  induced by a single additional choice: a unit imaginary octonion  $u \in S^6$ .

#### 3.1 The octonion splitting $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$

Choose a unit imaginary octonion  $u \in S^6 \subset \text{Im}(\mathbb{O})$ . Following the Fano convention, we set  $u = e_7$ .

*Remark 19* (Gap B, step 2). The choice of  $u$  is a second symmetry-breaking input, independent of the idempotent choice  $E_{11}$  (Remark 3). All unit imaginary octonions are conjugate under  $G_2 = \text{Aut}(\mathbb{O})$ ; the orbit is  $S^6 = G_2/\text{SU}(3)$ ,  $\dim(S^6) = 14 - 8 = 6$ . What selects a particular  $u$  is not addressed in this work.

The choice of  $u$  embeds  $\mathbb{C}$  into  $\mathbb{O}$ :

$$\mathbb{C} = \text{span}_{\mathbb{R}}\{1, e_7\} \subset \mathbb{O}. \quad (23)$$

The orthogonal complement of  $u$  in  $\text{Im}(\mathbb{O})$  is 6-dimensional:

$$W = u^\perp \cap \text{Im}(\mathbb{O}) = \text{span}_{\mathbb{R}}\{e_1, e_2, e_3, e_4, e_5, e_6\}, \quad \dim(W) = 6. \quad (24)$$

The space  $W$  inherits a complex structure from left multiplication by  $u$ :

$$J: W \rightarrow W, \quad J(w) = e_7 \cdot w. \quad (25)$$

To verify  $J^2 = -\text{Id}$ : for any  $w \in W$ , the left alternative identity  $x \cdot (x \cdot y) = (x \cdot x) \cdot y$  gives

$$J(J(w)) = e_7 \cdot (e_7 \cdot w) = (e_7 \cdot e_7) \cdot w = -w, \quad (26)$$

since  $e_7^2 = -1$ . The complex structure pairs the six real directions into three complex coordinates:  $(e_k, e_7 \cdot e_k)$  for  $k = 1, 2, 3$ . Consequently

$$\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3 \quad (\text{as real vector spaces: } 2 + 6 = 8), \quad (27)$$

where  $W \cong \mathbb{C}^3$  as a complex vector space (with  $J$  playing the role of multiplication by  $i$ ).

**Stabilizer.** The stabilizer of  $u$  in  $G_2 = \text{Aut}(\mathbb{O})$  is precisely the group that preserves the complex structure  $J$  on  $W \cong \mathbb{C}^3$ :

$$\text{SU}(3)_C := \text{Stab}_{G_2}(u) = \text{Stab}_{G_2}(e_7), \quad \dim(\text{SU}(3)_C) = 8. \quad (28)$$

This  $\text{SU}(3)_C$  acts on  $W \cong \mathbb{C}^3$  via the defining 3-dimensional representation and will be identified with the color group.

### 3.2 Cl(6) inside Cl(10)

The 6 directions of  $W$  become generators of a Clifford subalgebra inside  $\text{Cl}(10)$ . The bridge between the Peirce operators of Section 2 and the  $\text{Cl}(10)$  generators used here is as follows. The 9 rescaled Peirce operators  $\gamma_i = 2T_{a_i}$  ( $i = 1, \dots, 9$ ) generate  $\text{Cl}(9)$  inside  $\text{End}(V_{1/2}) = M_{16}(\mathbb{R})$  (Eq. (11)). After the complexification of Theorem 7,  $\text{End}(V_{1/2}^{\mathbb{C}}) = M_{16}(\mathbb{C})$  admits a 10th generator (arising from the structure group upgrade  $\text{Spin}(9) \rightarrow \text{Spin}(10)$ , Sec. 2.4), extending  $\text{Cl}(9)$  to  $\text{Cl}(10)$ . We denote the full set of  $\text{Cl}(10)$  generators by  $\Gamma_1, \dots, \Gamma_{10}$ , satisfying  $\{\Gamma_A, \Gamma_B\} = 2\delta_{AB}$ , where  $\Gamma_i = \gamma_i$  for  $i = 1, \dots, 9$  and  $\Gamma_{10}$  is the new generator from complexification. The 6 generators corresponding to the  $W$  directions define

$$\gamma_k := \Gamma_k, \quad k = 1, \dots, 6, \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij}, \quad (29)$$

generating the subalgebra

$$\text{Cl}(6) = \text{Alg}(\gamma_1, \dots, \gamma_6) \subset \text{Cl}(10). \quad (30)$$

The remaining 4 generators  $\Gamma_7, \Gamma_8, \Gamma_9, \Gamma_{10}$  correspond to: the  $u = e_7$  direction, two further traceless directions of  $h_2(\mathbb{O})$  (the real-diagonal traceless part of  $V_0$ ), and the 10th generator  $\Gamma_{10}$  arising from the complexification (Sec. 2.4).

*The Cl(6) subalgebra is not introduced ad hoc:* it is induced by the choice of complex structure  $u$ , which selects the 6 directions  $e_1, \dots, e_6 \in W = u^\perp \cap \text{Im}(\mathbb{O})$ . Concretely, the six imaginary octonion units  $e_k \in W$  act on  $V_{1/2} = \mathbb{O}^2$  as real-linear maps whose mutual anticommutativity ( $e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij}$  for  $i \neq j$ , from the octonion multiplication table) provides a  $\text{Cl}(0, 6)$  relation. The sign is absorbed by identifying the  $\text{Cl}(6)$  generators with the corresponding  $\text{Cl}(10)$  generators  $\gamma_k := \Gamma_k$  (Eq. (29)), which satisfy the positive convention  $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$  by construction. The two conventions are related by  $\gamma_k = i L_{e_k}$  (where  $L_{e_k}$  is left multiplication by  $e_k$ ), with the factor of  $i$  provided by the complexification of Theorem 7.

*Remark 20* (Fock space vs. spinor dimensions). The  $\text{Cl}(6)$  Fock space (from 3 creation operators  $a_j^\dagger$ ) is 8-dimensional ( $2^3$  states with particle number  $N = 0, 1, 2, 3$ ). The 16-dimensional  $S_{10}^+$  is a  $\text{Cl}(10)$  spinor, not a  $\text{Cl}(6)$  Fock space. The relationship is:  $\text{Cl}(6)$  acts on  $S_{10}^+$  via its embedding in  $\text{Cl}(10)$ , and the  $\text{Cl}(6)$  chirality operator  $\omega_6$  decomposes  $S_{10}^+ = S^+ \oplus S^-$  into two 8-dimensional eigenspaces, each carrying a  $\text{Cl}(6)$  Fock space structure. The 8 left-handed states (one generation of SM fermions) occupy one eigenspace; the 8 right-handed states occupy the other.

**Volume form.** The volume element of  $\text{Cl}(6)$  is

$$\omega_6 = \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6. \quad (31)$$

**Proposition 21** (Properties of  $\omega_6$ ). (a)  $\omega_6^2 = -1$ .

(b) *The chirality operator  $i\omega_6$  satisfies  $(i\omega_6)^2 = +1$ .*

(c)  $P = \frac{1}{2}(1 - i\omega_6)$  is an idempotent projector with  $\text{Tr}(P) = 16$  on the 32-dimensional Dirac spinor  $\Delta_{10}$  of  $\text{Cl}(10)$ .

*Proof.* (a) To compute  $\omega_6^2 = (\gamma_1 \cdots \gamma_6)^2$ , we move the second copy of each generator past the first. The number of transpositions is  $5 + 4 + 3 + 2 + 1 + 0 = 15 = 6 \cdot 5/2$ . Each transposition costs  $(-1)$  (from  $\{\gamma_i, \gamma_j\} = 0$  for  $i \neq j$ ), and each  $\gamma_k^2 = 1$ , giving

$$\omega_6^2 = (-1)^{15} \cdot 1 = -1. \quad (32)$$

Equivalently, the general formula  $\omega_n^2 = (-1)^{n(n-1)/2}$  gives  $(-1)^{15} = -1$  for  $n = 6$ .

(b)  $(i\omega_6)^2 = i^2 \cdot \omega_6^2 = (-1)(-1) = +1$ .

(c) Since  $(i\omega_6)^2 = 1$ , the element  $P = \frac{1}{2}(1 - i\omega_6)$  is idempotent:  $P^2 = \frac{1}{4}(1 - 2i\omega_6 + 1) = P$ . For the trace on the 32-dimensional Dirac spinor  $\Delta_{10} = S_{10}^+ \oplus S_{10}^-$ : the volume form  $\omega_6$  anticommutes with each  $\gamma_k$  (since  $\gamma_k\omega_6 = (-1)^5\omega_6\gamma_k = -\omega_6\gamma_k$ ), so its eigenvalues come in  $\pm$  pairs, giving  $\text{Tr}(\omega_6) = 0$ . Hence  $\text{Tr}(P) = \frac{1}{2}(32 - 0) = 16$ .  $\square$

The projector  $P$  selects a 16-dimensional subspace of  $\Delta_{10}$  corresponding to one generation of Standard Model fermions [19].

### 3.3 Pati-Salam breaking

The volume form  $\omega_6$  breaks  $\text{Spin}(10)$  to the subgroup that commutes with it. The 10 Clifford generators split into:

- **6 internal generators**  $\gamma_1, \dots, \gamma_6$  (the  $W$  directions from  $\text{Cl}(6)$ );
- **4 external generators**  $\Gamma_7, \Gamma_8, \Gamma_9, \Gamma_{10}$  (the complement).

The  $\binom{10}{2} = 45$  bivectors  $\Gamma_A\Gamma_B$  ( $A < B$ ) spanning  $\mathfrak{spin}(10)$  split into three classes based on their commutation with  $\omega_6$ :

**(i) Internal–internal:**  $\gamma_i\gamma_j$  for  $i, j \in \{1, \dots, 6\}$ ,  $i < j$ . Each factor anticommutes with  $\omega_6$ , so  $(\gamma_i\gamma_j)\omega_6 = \omega_6(\gamma_i\gamma_j)$ : *these commute with  $\omega_6$* . Count:  $\binom{6}{2} = 15$ . They generate  $\mathfrak{spin}(6) \cong \mathfrak{su}(4)$ .

**(ii) External–external:**  $\Gamma_A\Gamma_B$  for  $A, B \in \{7, \dots, 10\}$ ,  $A < B$ . Each external generator commutes with  $\omega_6$  (passing through 6 internal generators incurs  $(-1)^6 = +1$ ), so  $(\Gamma_A\Gamma_B)\omega_6 = \omega_6(\Gamma_A\Gamma_B)$ : *these commute with  $\omega_6$* . Count:  $\binom{4}{2} = 6$ . They generate  $\mathfrak{spin}(4) \cong \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ .

**(iii) Mixed:**  $\gamma_i\Gamma_A$  for  $i \in \{1, \dots, 6\}$ ,  $A \in \{7, \dots, 10\}$ . The internal factor anticommutes with  $\omega_6$  while the external factor commutes, so  $(\gamma_i\Gamma_A)\omega_6 = -\omega_6(\gamma_i\Gamma_A)$ : *these anticommute with  $\omega_6$*  and are not in the stabilizer. Count:  $6 \times 4 = 24$ .

**Stabilizer dimension check:**  $15 + 6 = 21$  (stabilizer) +  $24$  (complement) =  $45 = \dim \mathfrak{spin}(10)$ .  $\checkmark$

**Proposition 22** (Pati-Salam stabilizer). *The stabilizer of  $\omega_6$  in  $\text{Spin}(10)$  is the Pati-Salam group:*

$$\text{Stab}_{\text{Spin}(10)}(\omega_6) = \frac{\text{Spin}(6) \times \text{Spin}(4)}{\mathbb{Z}_2} \cong \frac{\text{SU}(4) \times \text{SU}(2)_L \times \text{SU}(2)_R}{\mathbb{Z}_2}, \quad \dim = 21. \quad (33)$$

**Representation decomposition.** Under Pati-Salam, the 16-dimensional Weyl spinor  $S_{10}^+$  (established in Section 2.4 as the complexified Peirce half-space  $V_{1/2}^{\mathbb{C}}$ ) decomposes as

$$\mathbf{16} \rightarrow (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}). \quad (34)$$

**Dimension check:**  $4 \times 2 \times 1 + 4 \times 1 \times 2 = 8 + 8 = 16$ .  $\checkmark$

The physical content of each factor:

- **(4, 2, 1):** 4 colors (3 quark + 1 lepton)  $\times$   $\text{SU}(2)_L$  doublet  $\times$   $\text{SU}(2)_R$  singlet = **left-handed fermions** ( $(u_L, d_L)$  in 3 colors +  $(\nu_L, e_L)$ ).
- **( $\bar{4}$ , 1, 2):**  $\bar{4}$  colors  $\times$   $\text{SU}(2)_L$  singlet  $\times$   $\text{SU}(2)_R$  doublet = **right-handed fermions** ( $(u_R, d_R)$  in 3 colors +  $(\nu_R, e_R)$ ).

This is the **LEFT (chiral) embedding:**  $\text{SU}(2)_L$  acts *only* on the  $(\mathbf{4}, \mathbf{2}, \mathbf{1})$  component, i.e. on left-handed fermions. Right-handed fermions are  $\text{SU}(2)_L$  singlets. This is precisely the chirality pattern of the Standard Model.

*Remark 23* (Left vs. diagonal embedding). In a diagonal embedding, a single  $\text{SU}(2)$  would act on *both* chiralities symmetrically, producing a non-chiral theory. Our construction excludes the diagonal embedding directly, without appeal to an external classification result. The  $\text{Cl}(6)$  volume form  $\omega_6$  defines a chirality grading ( $\omega_6 = -i \cdot (-1)^N$  on Fock states, Eq. (39)) that commutes with  $\text{SU}(4) = \text{Spin}(6)$  (the “internal” part, spanned by  $\gamma_1, \dots, \gamma_6$ ) but anti-commutes with the remaining four  $\text{Cl}(10)$  generators spanning  $\text{Spin}(4) = \text{SU}(2)_L \times \text{SU}(2)_R$ . The factor  $\text{SU}(2)_L$  is *by definition* the subgroup of  $\text{Spin}(4)$  preserving the  $\omega_6 = +i$  eigenspace; it therefore acts only on left-handed states. Right-handed states (the  $\omega_6 = -i$  eigenspace) are  $\text{SU}(2)_L$  singlets. This is a direct consequence of the Clifford algebra structure, not an imported result.

### 3.4 SM gauge group from Pati-Salam

The same complex structure  $u = e_7$  that produced  $\text{Cl}(6)$  further breaks  $\text{SU}(4)$ . The fundamental representation  $\mathbf{4}$  of  $\text{SU}(4) = \text{Spin}(6)$  acts on  $\mathbb{C}^4$ . The complex structure  $u$  distinguishes the lepton direction (associated with  $u$  itself) from the three quark/color directions:

$$\mathbb{C}^4 = \mathbb{C}_{\text{lepton}} \oplus \mathbb{C}_{\text{color}}^3. \quad (35)$$

The stabilizer of this decomposition in  $\text{SU}(4)$  is  $\text{SU}(3) \times \text{U}(1) \subset \text{SU}(4)$ , giving the Standard Model gauge group:

$$\text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)_Y \quad (36)$$

with hypercharge  $Y = (B-L) + 2J_3^R$ , where  $B-L$  comes from  $SU(4) \rightarrow SU(3) \times U(1)_{B-L}$  and  $J_3^R$  from  $SU(2)_R$ . This is the standard Gell-Mann–Nishijima normalization ( $Q = J_3^L + Y/2$ ), consistent with the quantum number assignments in Sec. 3.5.

**Dimension check:**  $\dim(SU(3)_C) + \dim(SU(2)_L) + \dim(U(1)_Y) = 8 + 3 + 1 = 12$ . ✓

**Full breaking chain:**

$$\text{Spin}(10) \xrightarrow{\omega_6} \frac{SU(4) \times SU(2)_L \times SU(2)_R}{\mathbb{Z}_2} \xrightarrow{u} SU(3)_C \times SU(2)_L \times U(1)_Y. \quad (37)$$

Both breaking steps are driven by the same input: the choice of complex structure  $u = e_7$  on  $\text{Im}(\mathbb{O})$ .

### 3.5 Quantum numbers of one generation

Following Furey [9], the Witt decomposition of  $\text{Cl}(6)$  defines creation and annihilation operators

$$a_j = \frac{1}{2}(\gamma_{2j-1} + i\gamma_{2j}), \quad a_j^\dagger = \frac{1}{2}(\gamma_{2j-1} - i\gamma_{2j}), \quad j = 1, 2, 3, \quad (38)$$

satisfying the canonical anticommutation relations  $\{a_i, a_j^\dagger\} = \delta_{ij}$ ,  $\{a_i, a_j\} = 0$ . The particle-number operator  $N = \sum_j a_j^\dagger a_j$  has eigenvalues 0, 1, 2, 3 on the  $2^3 = 8$  Fock states of the  $\text{Cl}(6)$  spinor. The volume form acts as

$$\omega_6 = -i \cdot (-1)^N, \quad (39)$$

so even- $N$  states have eigenvalue  $-i$  and odd- $N$  states have eigenvalue  $+i$ . The projector  $P = \frac{1}{2}(1 - i\omega_6)$  selects one chirality sector.

The Schwinger boson construction from  $\text{Cl}(4)$  Witt operators  $b_1, b_2$  gives the Cartan generators of  $SU(2)_L \times SU(2)_R$ :  $J_3^L = \frac{1}{2}(m_1 + m_2 - 1)$ ,  $J_3^R = \frac{1}{2}(m_1 - m_2)$ , where  $m_k = b_k^\dagger b_k$ .

The complete set of quantum number operators is:

$$B - L = 1 - \frac{2}{3}N, \quad Y = (B - L) + 2J_3^R, \quad Q = J_3^L + \frac{Y}{2}, \quad (40)$$

$$T_3^c = \frac{1}{2}(n_1 - n_2), \quad T_8^c = \frac{1}{2\sqrt{3}}(n_1 + n_2 - 2n_3). \quad (41)$$

All 16 SM states for one generation, with quantum numbers computed as explicit matrix eigenvalues [9, 19], are:

	Particle	$N$	$J_3^L$	$J_3^R$	$B-L$	$Y$	$Q$
$(\mathbf{4}, \mathbf{2}, \mathbf{1})$	$u_L(r, g, b)$	1	$+\frac{1}{2}$	0	$+\frac{1}{3}$	$+\frac{1}{3}$	$+\frac{2}{3}$
	$d_L(r, g, b)$	1	$-\frac{1}{2}$	0	$+\frac{1}{3}$	$+\frac{1}{3}$	$-\frac{1}{3}$
	$\nu_L$	3	$+\frac{1}{2}$	0	-1	-1	0
	$e_L$	3	$-\frac{1}{2}$	0	-1	-1	-1
$(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$	$u_R(r, g, b)$	1	0	$+\frac{1}{2}$	$+\frac{1}{3}$	$+\frac{4}{3}$	$+\frac{2}{3}$
	$d_R(r, g, b)$	1	0	$-\frac{1}{2}$	$+\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$
	$\nu_R$	3	0	$+\frac{1}{2}$	-1	0	0
	$e_R$	3	0	$-\frac{1}{2}$	-1	-2	-1

**Consistency checks:** 8  $SU(2)_L$  doublets ( $J_3^L = \pm\frac{1}{2}$ ) + 8 singlets ( $J_3^L = 0$ ); 4 color singlets (leptons) + 12 color triplets (quarks: 3 colors  $\times$  4);  $Q$  distribution  $\{+\frac{2}{3}: 6, -\frac{1}{3}: 6, 0: 2, -1: 2\}$  matches the Pati-Salam SM assignment;  $Y = (B - L) + 2J_3^R$  and  $Q = J_3^L + Y/2$  verified for all 16 states. Particle content:  $u_L \times 3, d_L \times 3, \nu_L, e_L, u_R \times 3, d_R \times 3, \nu_R, e_R$  — complete one-generation SM fermion content. Anomaly cancellation for one generation follows from the standard Spin(10) representation theory: the **16** is anomaly-free and does not require a separate check [19]. (The Fock-space particle number  $N$  is a Cl(6)-algebraic label, not a measure of compositeness: leptons occupy the  $N = 3$  sector because they are color singlets—all three color creation operators are “occupied”—while quarks at  $N = 1$  carry a single color index.)

*Remark 24* (Right-handed neutrino). The **16** of Spin(10) necessarily includes  $\nu_R$ , a right-handed neutrino that is a singlet under the full SM gauge group. This is a *prediction* shared by all SO(10)-based approaches. If  $\nu_R$  is realized in nature, it provides a natural candidate for Dirac neutrino masses. If it is absent (or acquires a large Majorana mass), the see-saw mechanism would apply. Either way, the construction accommodates current neutrino phenomenology.

## 4 Synthesis: One Choice, Two Consequences

The preceding two sections developed the Standard Model gauge group and its chiral representation by two apparently different routes: the  $F_4$  intersection (Part A, via Todorov-Drenska) and the Cl(6)/Pati-Salam chain (Part B, via Furey and Todorov). We now prove that both routes share a *single* algebraic input and that the second route provides a strict upgrade of the first.

### 4.1 The $F_4$ intersection route (Todorov-Drenska)

The choice of  $u \in S^6$  splits the octonions  $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$  (Section 3.1). This splitting induces a subgroup of  $F_4 = \text{Aut}(h_3(\mathbb{O}))$ : the elements that preserve the decomposition of each octonion entry into its  $\mathbb{C}$  and  $\mathbb{C}^3$  parts. The result, due to Todorov and Drenska [8] (see also Yokota [12]), is the maximal-rank subgroup

$$\frac{SU(3)_C \times SU(3)_J}{\mathbb{Z}_3} \subset F_4, \quad \dim = 8 + 8 = 16, \quad (42)$$

where  $SU(3)_C = \text{Stab}_{G_2}(u)$  acts on the  $\mathbb{C}^3$  factor of each octonion entry (the color group) and  $SU(3)_J = \text{Aut}(h_3(\mathbb{C}))$  acts on the Jordan algebra structure of the  $\mathbb{C}$ -valued part (a “Jordan flavor” group). The  $\mathbb{Z}_3$  quotient arises because the cube-root-of-unity centers of the two  $SU(3)$  factors intersect.

The SM gauge group emerges as the intersection of two stabilizers within  $F_4$ :

$$\text{Stab}_{F_4}(E_{11}) \cap \frac{SU(3)_C \times SU(3)_J}{\mathbb{Z}_3} = \text{Spin}(9) \cap \frac{SU(3)_C \times SU(3)_J}{\mathbb{Z}_3} = SU(3)_C \times SU(2) \times U(1) \quad (\text{at the Lie}) \quad (43)$$

The intersection works as follows:  $SU(3)_C \subset G_2 \subset Spin(9)$ , so it passes through the intersection intact. The  $SU(3)_J = \text{Aut}(h_3(\mathbb{C}))$  factor, intersected with  $\text{Stab}(E_{11})$  (fixing the first diagonal entry), reduces to  $U(2)_J = [SU(2) \times U(1)]/\mathbb{Z}_2$ .

**Dimension:**  $8 + 3 + 1 = 12 = \dim(SU(3)_C \times SU(2)_L \times U(1)_Y)$ . ✓

The equality is at the Lie algebra level: the intersection has  $\dim = 12$ , verified by explicit computation of the common generators in [8], Sec. 4. At the group level, discrete quotients (the  $\mathbb{Z}_3$  from  $SU(3)_C \times SU(3)_J/\mathbb{Z}_3$  and the  $\mathbb{Z}_2$  from  $[SU(2) \times U(1)]/\mathbb{Z}_2$ ) affect the global structure but not the gauge interactions.

*Remark 25* (No chirality from  $F_4$ ). The  $F_4$  route gives the gauge group but does *not* provide a chiral representation. The 27-dimensional representation of  $h_3(\mathbb{O})$  under  $F_4$  is real: there is no natural complex structure on **27** from the  $F_4$  perspective alone, and therefore no left/right distinction.

## 4.2 The single-input theorem

Both routes begin with the *same* decomposition of  $\text{Im}(\mathbb{O})$ :

$$\text{Im}(\mathbb{O}) = \text{span}\{u\} \oplus W, \quad W = u^\perp \cap \text{Im}(\mathbb{O}), \quad \dim(W) = 6. \quad (44)$$

**Theorem 26** (Single input). *Let  $u \in S^6 \subset \text{Im}(\mathbb{O})$  be a unit imaginary octonion, and let  $E_{11}$  be a rank-1 idempotent in  $h_3(\mathbb{O})$ . Then the single choice of  $u$  determines:*

- (i) **The  $F_4$  breaking.** *The splitting  $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$  induces  $[SU(3)_C \times SU(3)_J]/\mathbb{Z}_3 \subset F_4$ . Its intersection with  $Spin(9) = \text{Stab}_{F_4}(E_{11})$  gives  $SU(3)_C \times SU(2) \times U(1)$  at the Lie algebra level ( $\dim = 12$ ; [8], Sec. 4).*
- (ii) **The  $Cl(6)$  construction.** *The same  $W$  provides the 6 generators of  $Cl(6) \subset Cl(10)$ , whose volume form  $\omega_6$  breaks  $Spin(10) \rightarrow SU(4) \times SU(2)_L \times SU(2)_R$ . The same  $u$  further breaks  $SU(4) \rightarrow SU(3)_C \times U(1)_{B-L}$ , giving the SM gauge group with the LEFT chiral embedding.*

*No additional algebraic input beyond  $u$  and  $E_{11}$  is needed for either route.*

The two routes use the identical 6-dimensional subspace  $W$ —the same six directions  $e_1, \dots, e_6$  (in the Fano convention with  $u = e_7$ ). Their different outputs arise from different algebraic operations on  $W$ :

	<b>Route A</b> ( $F_4$ intersection)	<b>Route B</b> ( $Cl(6)$ /Pati-Salam)
Input	$u \in S^6$	same $u \in S^6$
From $W$	$J: W \rightarrow W$ defines $SU(3)_C = \text{Stab}_{G_2}(u)$	$\gamma_1, \dots, \gamma_6$ generate $Cl(6) \subset Cl(10)$
Intermediate	$[SU(3)_C \times SU(3)_J]/\mathbb{Z}_3 \subset F_4$	$\omega_6$ breaks $Spin(10) \rightarrow \text{Pati-Salam}$
Intersect/break	$\cap Spin(9)$	same $u$ breaks $SU(4) \rightarrow SU(3)_C \times U(1)$
Output	$SU(3)_C \times SU(2) \times U(1)$ ( $\dim = 12$ )	$SU(3)_C \times SU(2)_L \times U(1)_Y$ ( $\dim = 12$ )
Chirality	<b>none</b> (real <b>27</b> )	<b>LEFT</b> (complex <b>16</b> )

### 4.3 Group matching

We verify that the two routes produce the *same* gauge group factor by factor.

$SU(3)_C$ : In both routes,  $SU(3)_C$  is the group of transformations of  $W \cong \mathbb{C}^3$  preserving the complex structure  $J$  defined by  $u$ .

- Route A phrases this as  $\text{Stab}_{G_2}(u)$ : automorphisms of  $\mathbb{O}$  fixing  $u$ , which automatically preserve  $J(w) = u \cdot w$ .
- Route B phrases this as  $\text{Stab}_{SU(4)}(u)$ : the subgroup of  $SU(4) \cong \text{Spin}(6)$  (acting on the 6 directions of  $W$ ) that commutes with  $J$ . Since  $J$  is an orthogonal complex structure on  $W \cong \mathbb{R}^6$ , its stabilizer in  $SU(4)$  is  $SU(3) \times U(1)$ .

Both act on  $W \cong \mathbb{C}^3$  via the defining 3-dimensional representation.  $\therefore$  **Same**  $SU(3)_C$ .

$SU(2)$ :

- Route A:  $SU(2)$  from  $U(2)_J = \text{Stab}_{SU(3)_J}(E_{11})$ , the subgroup of the Jordan flavor  $SU(3)_J$  that preserves  $E_{11}$ . It acts on the 2nd and 3rd rows/columns of  $h_3(\mathbb{C})$ .
- Route B:  $SU(2)_L$  from  $\text{Spin}(4) = SU(2)_L \times SU(2)_R$ , the external part of  $\text{Stab}_{\text{Spin}(10)}(\omega_6)$ . It acts on the 4 external directions ( $\Gamma_7 - \Gamma_{10}$ ), which correspond to the  $V_0$  Peirce sector.

Both factors arise from the part of  $\text{Spin}(9)$  (or  $\text{Spin}(10)$  after complexification) acting on the “external” directions—the Peirce  $V_0$  sector rather than  $V_{1/2}$ . The  $\text{Cl}(6)$ /Pati-Salam route provides the additional information that this is  $SU(2)_L$  with a chiral action ( $\mathbf{16} \rightarrow (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$ ), whereas the  $F_4$  route sees only a generic  $SU(2)$  without left/right distinction.  $\therefore$  **Same**  $SU(2)$ .

$U(1)$ :

- Route A:  $U(1)$  from the center of  $U(2)_J = [SU(2) \times U(1)]/\mathbb{Z}_2$ . The generator acts as phase rotation on the off-diagonal octonion entries involving the first row/column of  $h_3(\mathbb{C}) \subset h_3(\mathbb{O})$ , distinguishing the observer’s slot ( $E_{11}$ ) from the remaining two.
- Route B:  $U(1)_Y$  with hypercharge  $Y = (B-L) + 2J_3^R$ . The generator  $B-L$  comes from the  $U(1)$  factor in  $SU(4) \rightarrow SU(3)_C \times U(1)_{B-L}$ , which is induced by the same  $u$  that splits  $\mathbb{O}$ .

Both generators live in the rank-4 Cartan subalgebra of  $F_4$  and are orthogonal to  $SU(3)_C$  and  $SU(2)$ . We do not independently derive the hypercharge formula; the explicit identification of the Route A generator with Route B hypercharge follows from Todorov and Drenska [8], who verify that the  $U(1)$  center of  $U(2)_J$  acts on the Peirce half-space  $V_{1/2}$  with the same eigenvalues as the hypercharge  $Y$  acts on  $S_{10}^+$ . This is a computation in [8], not just a rank count; our contribution is the framework that determines the algebraic input, not the hypercharge eigenvalue verification.  $\therefore$  **Same**  $U(1)$ .

Factor	Route A	Route B	Identification
$SU(3)_C$	$\text{Stab}_{G_2}(u)$	$\text{Stab}_{SU(4)}(u)$	Same: preserves $J: W \rightarrow W$ , acts on $W \cong \mathbb{C}^3$ via $\mathbf{3}$
$SU(2)$	$U(2)_J \cap \text{Spin}(9)$	$SU(2)_L \subset \text{Spin}(4)$	Same: acts on external ( $V_0$ ) directions
$U(1)$	Center of $U(2)_J$	$U(1)_Y$	Same: eigenvalues on $V_{1/2}$ match [8]
<b>Total</b>	dim = 12	dim = 12	$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$

## 4.4 The chiral upgrade theorem

**Theorem 27** (One choice, two consequences). *Let  $h_3(\mathbb{O})$  be the exceptional Jordan algebra. Assume:*

- (Gap A)  $h_3(\mathbb{O})$  is identified as the universe algebra via non-composability.
- (B1) An observer occupies a rank-1 idempotent  $e = E_{11}$ , reducing  $F_4 \rightarrow \text{Spin}(9)$  (all idempotents  $F_4$ -conjugate; necessary symmetry reduction).
- (B2) A complex structure  $u \in S^6 \subset \text{Im}(\mathbb{O})$  is selected (necessary symmetry reduction forced by Theorem 7 + Schur; all  $u$   $G_2$ -conjugate).

Then:

(a) **Gauge group.** *The intersection of  $\text{Spin}(9) = \text{Stab}_{F_4}(E_{11})$  with the  $u$ -preserving subgroup of  $F_4$  is the SM gauge group  $SU(3)_C \times SU(2) \times U(1)$  (dim = 12), verified by explicit computation [8].*

(b) **Chirality.** *The same  $u$  defines  $\text{Cl}(6) \subset \text{Cl}(10)$  whose volume form  $\omega_6$  selects the chiral (LEFT) embedding:  $\mathbf{16} \rightarrow (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$  under Pati-Salam, with  $SU(2)_L$  on left-handed fermions only.*

(c) **Upgrade.** *The  $\text{Cl}(6)$ /Pati-Salam route (b) provides a chiral representation for the same gauge algebra that the  $F_4$  intersection (a) provides without chirality. The chirality is not an additional postulate—it is a consequence of the same algebraic choice  $u$  that gives the gauge group.*

The comparison between the two routes is summarized in Table 2.

**The full chain:**

$$\text{self-modeling} \rightarrow h_3(\mathbb{O}) \xrightarrow{E_{11}} \text{Spin}(9) \xrightarrow{\text{C}^*\text{-complexification}} \text{Spin}(10) \xrightarrow{u \in S^6} SU(3)_C \times SU(2)_L \times U(1)_Y \text{ with LE} \quad (45)$$

## 5 Gap Analysis and Open Questions

The derivation chain of Table 1 involves three symmetry-breaking steps beyond the self-modeling axioms. Two of these (B1 and B2) are instances of *necessary symmetry reduction*

	$F_4$ intersection	$\text{Cl}(6)/\text{Pati-Salam}$
Gauge algebra	$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ (dim = 12)	Same (dim = 12)
Chirality	<b>None</b> — $F_4$ real, <b>27</b> real	<b>LEFT</b> — $\omega_6$ provides chirality
Representation	Acts on <b>27</b> (no chiral decomposition)	Acts on <b>16</b> (one generation, definite chirality)
Input	$E_{11} + u$	Same ( $E_{11} + u$ )
Cross-check value	Independent verification of gauge group	Strictly more informative (same algebra + chirality)

Table 2: Comparison of the two routes from the single input  $u \in S^6$ . The  $\text{Cl}(6)/\text{Pati-Salam}$  route provides a strict upgrade: it delivers the same gauge algebra plus chirality. The  $F_4$  route serves as an independent cross-check on the gauge group from a different algebraic framework.

in which the reduction is algebraically required and the direction is physically irrelevant (all choices are conjugate). The third (Gap A) is a genuine conditioning input. The complexification step (previously Gap C) has been closed by Theorem 7. This section presents a detailed analysis of each.

## 5.1 Conditional structure of the main result

The nine links of Table 1 fall into four classes according to their logical status.

### Proved (within this work and companions):

- **L1:** Self-modeling forces  $M_n(\mathbb{C})^{\text{sa}}$  [1].
- **L5:** Given complexification (L4), the upgrade  $F_4 \rightarrow E_6$  and the decomposition  $\mathbf{27} \rightarrow \mathbf{1} \oplus \mathbf{10} \oplus \mathbf{16}$  follow from standard representation theory (Sec. 2).
- **L7–L9:** Given  $\text{Spin}(10)$  and  $u \in S^6$ , the  $\text{Cl}(6)$  chirality,  $F_4$  intersection, and their synthesis are proved (Secs. 3–4).

### Derived (formerly Gap C):

- **L4:** The complexification step is now proved (Theorem 7): the continuous functional calculus of the observer’s  $\text{C}^*$ -algebra, applied to the Peirce operators  $T_a$  (which have eigenvalue  $-\frac{1}{2} < 0$ ), produces complex output:  $\sqrt{T_a} T_b \sqrt{T_a} = (i/2) T_b$ . The measurement algebra exits  $M_{16}(\mathbb{R})$  and generates  $M_{16}(\mathbb{C}) = \text{Cl}(9, \mathbb{C})$ .

### Conditioning input (Gap A):

- **L2 (Gap A):** Non-composability identifies  $h_3(\mathbb{O})$  as the universe algebra. Everything downstream (L3–L9) takes place within  $h_3(\mathbb{O})$ . This is the only genuine conditioning input: it requires Level IV mathematical realism (Remark 2).

### Necessary symmetry reduction (B1, B2):

- **L3 (B1):** Observer occupies a rank-1 idempotent  $E_{11}$ , breaking  $F_4 \rightarrow \text{Spin}(9)$ . All rank-1 idempotents are  $F_4$ -conjugate; the breaking is forced by the observer’s existence and the direction is gauge.
- **L6 (B2):** Observer selects  $u \in S^6$ , breaking  $G_2 \rightarrow \text{SU}(3)_C$ . Theorem 7 establishes complexification; Schur’s lemma forbids equivariant complexification; therefore the observer *must* select some  $u$  (Remark 12). All  $u$  are  $G_2$ -conjugate; the breaking is forced and the direction is gauge.

We use “necessary symmetry reduction” to mean that the algebraic structure *requires* a symmetry-reducing choice (a representative within a conjugacy class), and all choices are physically equivalent. No effective potential, order parameter, or Goldstone analysis is invoked: the reduction is algebraic, not dynamical. Whether a dynamical mechanism underlies these reductions is connected to Gap SA and deferred to future work. The terminology is chosen to distinguish this phenomenon from dynamical spontaneous symmetry breaking, which involves vacuum degeneracy and Goldstone modes.

**The strongest conditional result:**

If the universe algebra is  $h_3(\mathbb{O})$  (Gap A), then the existence of a self-modeling observer forces—via two instances of necessary symmetry reduction whose directions are physically irrelevant—the SM gauge group  $\text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)_Y$  and its chiral (LEFT) one-generation representation.

Gap A (Level IV realism + non-composability) is the *only* conditioning input beyond the self-modeling axioms. The two symmetry reductions (B1, B2) are required by the algebraic structure: the observer must occupy some idempotent (B1) and must complexify non-equivariantly (B2), but all choices produce identical physics ( $F_4$ -conjugacy for B1;  $G_2$ -conjugacy for B2). The complexification step (previously Gap C) is derived from the observer’s  $C^*$ -algebra continuous functional calculus (Theorem 7).

## 5.2 Gap register

Table 3 classifies the five gaps identified in the derivation chain, ordered by their logical role and severity.

**Severity justification:**

- **B1 (NSR):** The observer must occupy *some* rank-1 idempotent to exist as a Peirce subsystem of  $h_3(\mathbb{O})$ ; this reduces  $F_4 \rightarrow \text{Spin}(9)$ . All rank-1 idempotents are  $F_4$ -conjugate (orbit =  $\mathbb{O}P^2$ ), so the physics is identical regardless of which idempotent is selected. This is a necessary symmetry reduction: the reduction is forced by the observer’s existence; the direction is a gauge choice.
- **B2 (NSR):** Theorem 7 establishes complexification of  $V_{1/2}$ . Schur’s lemma forbids equivariant complexification ( $\text{End}_{\text{Spin}(9)}(S_9) = \mathbb{R}$ , since  $9 \bmod 8 = 1$  gives real type by Bott periodicity). Therefore the observer *must* select some  $u \in S^6$ —there is no equivariant alternative. The selection mechanism is the CFC computation  $\sqrt{T_a}(\cdot)\sqrt{T_a}$ : every instance of the self-modeling cycle that accesses  $V_{1/2}$  applies this operation for

a specific  $T_a$ , and this act *is* the choice of  $u$  (Remark 12). No external dynamics is needed; the self-modeling cycle is the dynamics. All  $u$  are  $G_2$ -conjugate, so the physics is identical regardless of which  $u$  is selected. This is a necessary symmetry reduction:  $G_2 \rightarrow \text{SU}(3)_C$ , with the reduction required by the algebra and the direction a gauge choice. The situation is structurally analogous to the direction of the electroweak vacuum (cf. [18] for an explicit  $G_2 \rightarrow \text{SU}(3)$  model on  $S^6$ ), though we do not invoke an effective potential here.

- **A (HIGH):** The identification of  $h_3(\mathbb{O})$  as the universe algebra rests on the JvNW classification (standard mathematics) combined with Level IV mathematical realism: if an algebra can form composites, then composites of it exist, making it a subsystem rather than the universe. By contrapositive, the universe’s algebra must be non-composable. L4 realism is the program’s foundational premise (not a new assumption), but it is doing substantive work here. Without it, non-composability would be merely *permitted*, not *required*.
- **C (CLOSED):** The complexification step (L4) is proved by Theorem 7. The observer’s  $C^*$ -algebra continuous functional calculus, applied to Peirce operators with negative eigenvalues, necessarily produces imaginary coefficients that extend the measurement algebra from  $M_{16}(\mathbb{R})$  to  $M_{16}(\mathbb{C})$ . This result is compatible with the impossibility of  $\text{Spin}(9)$ -equivariant complexification (Schur’s lemma): the observer-induced complexification is non-equivariant, depending on which  $T_a$  is measured first (Remark 11).
- **Gen (LOW for this paper; HIGH for the program):** The present work addresses one generation of fermions. The question of why there are exactly three generations is explicitly out of scope. For the research program as a whole, however, this gap is HIGH: producing three generations with the observed mass hierarchy and CKM/PMNS mixing is among the deepest open problems in particle physics, and no algebraic approach has solved it.
- **SA (LOW for this paper):** Computing the spectral action (the GR + SM Lagrangian) from the algebraic data established here is a separate research direction, deferred to future work.

### 5.3 Structural independence and the reduction cascade

The two symmetry reductions (B1 and B2) are *structurally independent*: each is forced by the algebra, and neither constrains the other’s direction.

**B1 does not constrain B2.** The stabilizer  $\text{Spin}(9) = \text{Stab}_{F_4}(E_{11})$  contains  $G_2 = \text{Aut}(\mathbb{O})$ , which acts transitively on  $S^6$ . Therefore, after fixing  $E_{11}$ , all complex structures  $u \in S^6$  remain equivalent—the first reduction does not select a preferred direction for the second.

**B2 does not constrain B1.** The  $u$ -preserving subgroup  $[\text{SU}(3)_C \times \text{SU}(3)_J]/\mathbb{Z}_3$  acts transitively on the rank-1 idempotents of  $h_3(\mathbb{C}) \subset h_3(\mathbb{O})$ , but not on all rank-1 idempotents of  $h_3(\mathbb{O})$ . Fixing  $u$  restricts the orbit of idempotents from  $\mathbb{O}P^2$  (dim 16) to a smaller manifold, but does not determine a unique  $E_{11}$ .

**The reduction cascade.** The two reductions form a cascade: the observer’s existence reduces  $F_4 \rightarrow \text{Spin}(9)$  (B1), and the observer’s probing of  $V_{1/2}$  reduces  $G_2 \rightarrow \text{SU}(3)_C$  (B2). Each step is algebraically required (the observer must exist; it must probe  $V_{1/2}$ ) and each direction is physically irrelevant (all idempotents are  $F_4$ -conjugate; all  $u$  are  $G_2$ -conjugate). The algebraic mechanism driving the cascade is the observer’s  $C^*$ -nature—inherited from the self-modeling cycle of Paper 5—not an external potential or Lagrangian. The situation is structurally analogous to the Higgs mechanism (the symmetry must reduce; the direction is gauge), but no effective potential is invoked here.

**Gap A is logically upstream.** Without the identification of  $h_3(\mathbb{O})$  as the universe algebra, neither the rank-1 idempotent  $E_{11}$  nor the complex structure  $u$  is defined. Gap A is a prerequisite for B1 and B2; closing A would not affect the reduction steps, but opening A would render them moot.

**Gen and SA are downstream/separate.** The generation question (why 3 families) and the spectral action computation do not affect the gauge group + chirality result established in this paper. They represent directions for extending the result, not challenges to its validity within its stated scope.

## 6 Discussion

### 6.1 The research arc: Papers 5 and 7

This paper extends the self-modeling program begun in Paper 5.

**Paper 5 [1] — Quantum mechanics from self-modeling.** The requirement that a finite-dimensional system contains a faithful model of itself, updated through its boundary, forces quantum mechanics: the state space is isomorphic to  $M_n(\mathbb{C})^{sa}$  (the self-adjoint part of a matrix  $C^*$ -algebra). The result is unconditional: the only input is the self-modeling axiom itself.

**Paper 7 (this work) — Standard Model chirality from  $h_3(\mathbb{O})$ .** A  $C^*$ -observer probing the exceptional Jordan algebra  $h_3(\mathbb{O})$  obtains the SM gauge group  $\text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)_Y$  with the LEFT chiral representation, from a single algebraic choice: a complex structure  $u \in S^6 \subset \text{Im}(\mathbb{O})$ . The additional inputs beyond self-modeling are the identification of  $h_3(\mathbb{O})$  as the universe algebra (Gap A) and two symmetry-breaking choices (B1 and B2).

Paper 5 requires no inputs beyond the self-modeling axiom; Paper 7 requires the  $h_3(\mathbb{O})$  identification (Gap A) plus two instances of necessary symmetry reduction ( $E_{11}$  and  $u$ ) whose directions are physically irrelevant. The increasing conditionality reflects the increasing specificity of the target: quantum mechanics is generic to all self-modeling systems, while the specific SM gauge group requires exceptional algebraic structure.

### 6.2 Relationship to prior work

Several groups have explored the connection between exceptional algebraic structures and the Standard Model. We compare our approach with four lines of work.

**Todorov and Drenska [8, 19].** Todorov and Drenska showed that the  $F_4$  intersection—the subgroup of  $F_4$  that preserves the decomposition  $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$  induced by a complex structure, intersected with  $\text{Spin}(9) = \text{Stab}_{F_4}(E_{11})$ —contains the SM gauge group  $\text{SU}(3)_C \times \text{SU}(2) \times \text{U}(1)$ . *Our contribution:* We reproduce this result (Sec. 4.1) and additionally (i) derive the complexification step from the  $C^*$ -observer nature (Theorem 7) rather than assuming it, (ii) show that the same algebraic input  $u$  that produces the  $F_4$  route simultaneously produces the  $\text{Cl}(6)$  chirality (Theorem 26), and (iii) embed the result in the self-modeling framework (Paper 5).

**Furey [9].** Furey showed that the Clifford algebra  $\text{Cl}(6)$ , via the Witt decomposition, produces automatic chirality for one generation of SM fermions: the volume form  $\omega_6$  separates left-handed from right-handed states. *Our contribution:* We reproduce this result (Sec. 3) and additionally (i) trace the  $\text{Cl}(6)$  subalgebra to the choice of complex structure  $u$  on  $\text{Im}(\mathbb{O})$  (it is induced, not postulated), (ii) show that this same  $u$  produces the  $F_4$  route as well (the single-input synthesis of Sec. 4), and (iii) derive the complexification (Theorem 7) that provides the 16-dimensional complex representation on which  $\text{Cl}(6)$  acts.

**Boyle [10].** Boyle studied the complexified exceptional Jordan algebra  $h_3^{\mathbb{C}}(\mathbb{O})$  and its structure group  $E_6$ , identifying the Weyl spinor  $S_{10}^+$  and the decomposition  $\mathbf{27} = \mathbf{1} \oplus \mathbf{10} \oplus \mathbf{16}$  under  $\text{Spin}(10)$ . *Our contribution:* We arrive at the same decomposition (Sec. 2.4) but *derive* the complexification from the  $C^*$ -observer nature (Sec. 2.3) rather than taking it as a mathematical starting point. The end result is the same identification of  $V_{1/2}^{\mathbb{C}} = S_{10}^+$ ; the logical route differs: Boyle assumes complexification, we derive it from self-modeling.

**Connes, Chamseddine, and Marcolli [2].** The spectral triple approach of Connes and collaborators derives the SM Lagrangian from noncommutative geometry with the finite algebra  $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ . The choice of this direct-sum algebra is an input matched to observation. As noted in the introduction, simple  $M_n(\mathbb{C})$  algebras cannot reproduce the SM within this framework—this is the Barrett no-go result [3] that motivates our turn to  $h_3(\mathbb{O})$ . *Our contribution:* The present work provides the algebraic input—gauge group and chiral representation—from the  $h_3(\mathbb{O})$  structure. The spectral action computation (deriving the full GR + SM Lagrangian from an  $h_3(\mathbb{O})$ -based spectral triple) is identified as Gap SA and deferred to future work. If successful, this would provide a non-perturbative completion of the CCM program with the finite algebra determined by the self-modeling principle rather than chosen by hand.

**Dubois-Violette [14].** Dubois-Violette was among the earliest to connect  $h_3(\mathbb{O})$  to the Standard Model, studying the algebra’s internal symmetries and their relation to gauge groups. *Our contribution:* We provide the self-modeling framework (Paper 5) that motivates  $h_3(\mathbb{O})$  as the universe algebra via non-composability, rather than adopting it as a mathematical starting point.

### 6.3 Novelty delineation

The individual algebraic components of our derivation are established results. The contribution of this paper lies in four new connections.

1. **Complexification traced to C\*-observer nature (new).** We prove that the upgrade  $\text{Spin}(9) \rightarrow \text{Spin}(10)$  and  $F_4 \rightarrow E_6$  is produced by the *combination* of the observer’s C\*-algebra nature (Paper 5 [1]) and the negative-spectrum Peirce operators of  $h_3(\mathbb{O})$ . The observer’s C\*-algebra continuous functional calculus, applied to these operators, necessarily produces complex coefficients, extending the measurement algebra from  $M_{16}(\mathbb{R})$  to  $M_{16}(\mathbb{C})$  (Theorem 7). Neither input alone suffices: a real-algebraic observer would have no complex functional calculus; an operator with non-negative spectrum would have a real square root. This replaces the bare assumption of complexification in [10] with a conditional derivation from identified inputs.
2. **Cl(6) chirality and  $F_4$  intersection traced to the same input  $u$  (new).** The literature treats the Cl(6)/Furey route and the  $F_4$ /Todorov–Drenska route as separate approaches. Theorem 26 proves they share the same algebraic input ( $u \in S^6$ ) and produce the same gauge group, with the Cl(6) route additionally providing chirality (Theorem 27).
3. **Symmetry reductions shown to be algebraically necessary (new).** The two symmetry-reducing steps ( $E_{11}$  and  $u$ ) are shown to be instances of necessary symmetry reduction: the observer *must* occupy an idempotent (B1) and *must* complexify non-equivariantly (B2, via Theorem 7 + Schur’s lemma), while all choices are conjugate ( $F_4$  for B1,  $G_2$  for B2). This reduces the conditioning inputs from three to one (Gap A).
4. **Complete chain from self-modeling to chiral SM (new).** No prior work connects the self-modeling axioms (Paper 5) through  $h_3(\mathbb{O})$  complexification (Part A) to the chiral SM representation (Part B) in a single derivation chain with explicit gap identification (Table 1, Table 3).

The established components on which we build include: the JvNW classification of formally real Jordan algebras [4, 6], the structure of  $h_3(\mathbb{O})$  and its automorphism group  $F_4$  [7, 12], the  $\text{Cl}(6) \rightarrow \text{Pati-Salam} \rightarrow \text{SM}$  chain [9, 19], the  $F_4$  intersection route [8], and the  $h_3^{\mathbb{C}}(\mathbb{O})/E_6$  analysis [10].

## 6.4 Outlook

We identify four concrete directions for extending the present result.

**1. The reduction cascade: B1 and B2 as dynamical symmetry breaking.** We have identified B1 and B2 as necessary symmetry reductions (Sec. 5.1). A natural extension is to investigate whether the spectral action on an  $h_3(\mathbb{O})$ -based spectral triple generates effective potentials on the orbit spaces  $\mathbb{O}P^2$  (for B1) and  $S^6$  (for B2) that make the reductions dynamical rather than purely algebraic. An explicit  $G_2 \rightarrow \text{SU}(3)$  symmetry-breaking model on  $S^6$  with a quartic potential has been constructed by Ferrara et al. [18]; deriving such a potential from the spectral action would connect the reduction cascade to Gap SA. Note, however, that no such potential is *needed* for the present result: all choices are conjugate, so the physics is direction-independent.

**2. Generation structure.** The “3” in  $h_3(\mathbb{O})$  (the algebra consists of  $3 \times 3$  matrices) is suggestive: three off-diagonal octonion entries might correspond to three generations.

However, no mechanism producing three copies of the **16** representation has been established. Furey [9] proposed a route via the full octonion algebra  $\mathbb{O}$  acting on  $\text{Cl}(6)$ ; Boyle [10] explored triality. Making this connection rigorous within the self-modeling framework is an open challenge.

**3. Spectral action.** Computing the full GR + SM Lagrangian from the algebraic data established here—the  $h_3(\mathbb{O})$ -based spectral triple with  $\text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)_Y$  gauge group and chiral representation—is the most direct extension. The spectral action principle [2] provides the framework; the challenge is constructing the Dirac operator and evaluating the action for the exceptional algebra, which is non-associative.

**4. Connection to other exceptional structures.** The exceptional groups  $G_2, F_4, E_6, E_7, E_8$  appear throughout this work. Their role in the exceptional magic square [7] and in higher-dimensional extensions (e.g.,  $E_8 \times E_8$  in string theory) suggests further structure beyond the SM that the self-modeling framework might illuminate.

## 6.5 What this paper does not claim

To prevent misreading, we state explicitly what the paper does *not* claim.

1. **Not a derivation of the Standard Model.** The paper extracts a one-generation chiral gauge representation from  $h_3(\mathbb{O})$  given one conditioning input (Gap A) and two instances of necessary symmetry reduction (B1, B2), plus the complexification derived in Theorem 7. It does not derive the SM Lagrangian, the Higgs sector, Yukawa couplings, the mass hierarchy, CKM/PMNS mixing, or anomaly cancellation.
2. **The complexification is proved but non-equivariant.** Theorem 7 derives complexification from the observer’s C\*-algebra functional calculus. The resulting complex structure depends on which Peirce operator  $T_a$  the observer measures first, corresponding to the choice of  $u \in S^6$  (B2). No Spin(9)-equivariant complexification exists (Remark 11). This non-equivariance is a feature, not a defect: it is what makes B2 an instance of a necessary symmetry reduction rather than a free choice (Remark 12).
3. **The symmetry reductions are physically irrelevant in direction but structurally essential.** The SM-like gauge structure requires the algebraic data ( $E_{11}$  and  $u$ ), but all choices are conjugate ( $F_4$ -conjugate for  $E_{11}$ ;  $G_2$ -conjugate for  $u$ ). The paper’s contribution is to show that (i) the breaking is forced by the algebra, (ii) the direction is gauge, and (iii) the single choice  $u$  produces a *coherent* structure (one input, two consequences).
4. **One generation only.** No mechanism producing three generations of fermions from  $h_3(\mathbb{O})$  has been established.
5. **No dynamics.** The spectral action computation (Gap SA) is deferred to future work.
6. **Interpretive framework inherited from Paper 5.** The identification of the sequential product  $a \& b = \sqrt{a} b \sqrt{a}$  with physical measurement composition is derived in Paper 5 for effects; the present work inherits the C\*-algebra structure that Paper 5

establishes and applies its continuous functional calculus to the Peirce operators of  $h_3(\mathbb{O})$ . The CFC computation  $\sqrt{T_a} T_b \sqrt{T_a}$  coincides with the sequential product on effects (Lemma 6) and is the canonical extension of that operation to all self-adjoint elements via the C\*-algebra structure.

## Acknowledgments

[Acknowledgments placeholder.]

## References

- [1] Bryan Ehrlich. Quantum mechanics from self-modeling. Paper 5 in this series, 2026.
- [2] Ali H. Chamseddine, Alain Connes, and Matilde Marcolli. Gravity and the standard model with neutrino mixing. *Adv. Theor. Math. Phys.*, 11:991–1089, 2007. doi: 10.4310/ATMP.2007.v11.n6.a3.
- [3] John W. Barrett. A Lorentzian version of the non-commutative geometry of the Standard Model of particle physics. *J. Math. Phys.*, 48:012303, 2007. doi: 10.1063/1.2408400.
- [4] Pascual Jordan, John von Neumann, and Eugene Wigner. On an algebraic generalization of the quantum mechanical formalism. *Ann. Math.*, 35:29–64, 1934. doi: 10.2307/1968117.
- [5] Howard Barnum, Matthew A. Graydon, and Alexander Wilce. Composites and categories of euclidean jordan algebras. *Quantum*, 4:359, 2020. doi: 10.22331/q-2020-11-08-359.
- [6] A. Adrian Albert. On a certain algebra of quantum mechanics. *Ann. Math.*, 35:65–73, 1934. doi: 10.2307/1968118.
- [7] John C. Baez. The octonions. *Bull. Amer. Math. Soc.*, 39:145–205, 2002. doi: 10.1090/S0273-0979-01-00934-X.
- [8] Ivan Todorov and Svetla Drenska. Composition algebras, exceptional Jordan algebra and related groups. 2018.
- [9] Cohl Furey. Three generations, two unbroken gauge symmetries, and one eight-dimensional algebra. *Phys. Lett. B*, 785:84–89, 2018. doi: 10.1016/j.physletb.2018.08.032.
- [10] Latham Boyle. The standard model, the exceptional Jordan algebra, and triality. 2020.
- [11] Kirill Krasnov. Pure spinors and the Standard Model. 2025.
- [12] Ichiro Yokota. *Exceptional Lie Groups*. arXiv preprint, 2009.

- [13] Jogesh C. Pati and Abdus Salam. Lepton number as the fourth “color”. *Phys. Rev. D*, 10:275–289, 1974. doi: 10.1103/PhysRevD.10.275.
- [14] Michel Dubois-Violette. Exceptional quantum geometry and particle physics. *Nucl. Phys. B*, 912:426–449, 2016. doi: 10.1016/j.nuclphysb.2016.04.018.
- [15] Max Tegmark. The mathematical universe. *Found. Phys.*, 38:101–150, 2008. doi: 10.1007/s10701-007-9186-9.
- [16] Nicholas J. Higham. *Functions of Matrices: Theory and Computation*. Society for Industrial and Applied Mathematics, 2008. doi: 10.1137/1.9780898717778.
- [17] Stan Gudder and Richard Greechie. Sequential products on effect algebras. *Rep. Math. Phys.*, 49:87–111, 2002. doi: 10.1016/S0034-4877(02)80007-6.
- [18] Alessio Ferrara, Florian Ferretti, Marcus Sperling, and George Zoupanos. An exceptional  $G(2)$  extension of the Standard Model from the correspondence with Clifford algebras. *Sci. Rep.*, 11:22528, 2021. doi: 10.1038/s41598-021-01814-1.
- [19] Ivan Todorov. Superconnection in the spin factor approach to particle physics. 2022.

Table 3: Gap register for the derivation chain. Severity reflects how the gap affects the chain if left open. “What would close it” indicates the type of argument or result needed, not a claim that closure is imminent.

Gap	Description	Nature	Severity
B1	Rank-1 idempotent $E_{11}$ choice	Necessary symmetry reduction: the observer must occupy <i>some</i> rank-1 idempotent to exist as a Peirce subsystem. All rank-1 idempotents are $F_4$ -conjugate (orbit = $\mathbb{O}P^2$ , dim 16). Reduction is required; direction is gauge.	NS
B2	Complex structure $u \in S^6$ choice	Necessary symmetry reduction: Theorem 7 establishes complexification; Schur’s lemma forbids equivariant complexification. Observer <i>must</i> select some $u$ . All $u$ conjugate under $G_2$ (orbit = $S^6 = G_2/SU(3)$ , dim 6). Reduction is required; direction is gauge.	NS
A	Non-composability $\rightarrow h_3(\mathbb{O})$	Standard math (JvNW [4] + BGW [5]) + Level IV realism (composable algebra $\Rightarrow$ composites exist $\Rightarrow$ not the universe). The L4 step is the program’s foundational premise, not a new assumption, but it is doing real work here.	HIC
Gen	Why 3 generations	Open question. The “3” in $h_3(\mathbb{O})$ is suggestive, but no mechanism producing 3 copies of the <b>16</b> representation has been established.	LOW (paper) HIGH (program)
C	Complexification (L4)	CLOSED (Theorem 7). The observer’s C*-algebra CFC, applied to $T_a$ (eigenvalue $-1/2 < 0$ ), produces complex output: $\sqrt{T_a} T_b \sqrt{T_a} = (i/2) T_b$ . Measurement algebra exits $M_{16}(\mathbb{R})$ into $M_{16}(\mathbb{C})$ .	CLOSED (was: F)
SA	Spectral action computation	Deferred: the current work establishes the algebraic input (gauge group + chirality) but not the dynamics (GR + SM Lagrangian).	LO