Chirality from $h_3(\mathbb{O})$: Standard Model Gauge Group and Chiral Representation from Self-Modeling via Exceptional Jordan Algebra

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Self-modeling—the requirement that a physical system contains a faithful model of itself—forces quantum mechanics with $M_n(\mathbb{C})^{sa}$ state spaces [1]. We argue that the exceptional Jordan algebra $h_3(\mathbb{O})$ is the unique candidate for the "universe algebra" via its non-composability (it cannot be a subsystem of a larger system). A C*-observer probing $h_3(\mathbb{O})$ induces a Peirce decomposition under a rank-1 idempotent E_{11}, and the observer's C*-algebra nature forces complexification of the Peirce half-space, upgrading the symmetry from Spin(9) to Spin(10) acting on a 16-dimensional complex Weyl spinor S^+_{10}. A single additional algebraic choice—a complex structure $u \in S^6 \subset \mathrm{Im}(\mathbb{O})$—simultaneously determines both the Standard Model gauge group $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ via F_4 intersection and the chiral (left-handed) fermion representation via the Cl(6) volume form. The result is conditional on three inputs: the identification of $h_3(\mathbb{O})$ as the universe algebra, the choice of rank-1 idempotent, and the choice of complex structure. We present the complete 9-link chain from self-modeling axioms to the chiral Standard Model representation, with explicit classification of each link as proved, established, or an open gap.

I. INTRODUCTION

The Standard Model gauge group $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ with its chiral fermion representation—left-handed doublets and right-handed singlets under $\mathrm{SU}(2)_L$—is the empirical target of any program that seeks to derive particle physics from deeper principles. In a companion paper [1], we showed that *self-modeling*—the requirement that a physical system contains a faithful model of itself, updated through its boundary—forces quantum mechanics: the state space of any self-modeling system is isomorphic to $M_n(\mathbb{C})^{sa}$, the self-adjoint part of a matrix C*-algebra. A second companion paper [2] showed that self-modeling locality on a lattice yields Einstein's equations at long wavelengths.

These results leave open a structural question. Simple $M_n(\mathbb{C})$ algebras produce the gauge group $\mathrm{U}(n)$ via the spectral triple construction of Connes [3]—not the Standard Model. To obtain the SM gauge group within the Connes–Chamseddine–Marcolli framework, the finite algebra must be taken as a direct sum, $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, an input chosen to match observation. This is a *structural obstruction*: self-modeling forces simple matrix algebras, but simple matrix algebras cannot reproduce the SM within the spectral triple approach. The obstruction is not a computational gap but a no-go result [4]: no Dirac operator on $M_n(\mathbb{C})$ yields the SM gauge group.

This motivates turning to a different algebraic structure: the exceptional Jordan algebra $h_3(\mathbb{O})$, the algebra of 3×3 Hermitian matrices over the octonions $\mathbb{O}$. The exceptional Jordan algebra occupies a singular position in the Jordan–von Neumann–Wigner classification [5]: it is the unique finite-dimensional formally real Jordan algebra that is *non-composable*—it cannot participate in any composite with a nontrivial system [6]. Under Level IV mathematical realism, if an algebra can form composites then composites of it exist, making it a subsystem of a larger structure. The universe, by definition, is not a subsystem; therefore its algebra must be non-composable. This makes $h_3(\mathbb{O})$ the unique candidate for the "universe algebra" whose internal structure should encode the particle content [7, 8].

The connection between $h_3(\mathbb{O})$ and the Standard Model has been explored from several directions. Todorov and Drenska [9] showed that the $F_4 = \mathrm{Aut}(h_3(\mathbb{O}))$ intersection with the stabilizer of a rank-1 idempotent gives the SM gauge group. Furey [10] showed that the Clifford algebra Cl(6), via the Witt decomposition, produces automatic chirality for a single generation of SM fermions. Boyle [11] studied the complexification $h_3^{\mathbb{C}}(\mathbb{O})$ and its connection to E_6. Krasnov [12] explored the role of pure spinors and complex structures on $\mathbb{O}^2$.

The present paper contributes three new results to this program:

1. **Complexification derived, not assumed.** The complexification $\mathrm{Spin}(9) \to \mathrm{Spin}(10)$ is forced by the C*-algebra nature of the observer (from self-modeling [1]), rather than imposed by hand as in [11]. This is Part A of the argument (Sec. II).
2. **One choice, two consequences.** The F_4 intersection route (Todorov–Drenska) and the Cl(6)/Pati-Salam route (Furey) are traced to a *single* algebraic input: the choice of a complex structure $u \in S^6 \subset \mathrm{Im}(\mathbb{O})$. Both routes produce the same gauge algebra $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$; the Cl(6) route additionally provides the chiral representation.
3. **Complete chain with explicit gaps.** No prior work connects the self-modeling axioms through $h_3(\mathbb{O})$ to the chiral SM in a single chain with hon-

est gap identification. Table I presents the 9-link chain from self-modeling to chiral fermions.

A. The derivation chain

Table I presents the complete logical chain from self-modeling axioms to the chiral Standard Model representation. The chain consists of nine links, each classified by source and status. Two links are explicit *gaps*—symmetry-breaking inputs that are not derived from the self-modeling framework. One link is *established* by standard mathematics combined with a physically motivated but separate argument. The remaining six links are *proved* within this work and its companions.

The result is conditional on three inputs: (i) the identification of $h_3(\mathbb{O})$ as the universe algebra via non-composability (Gap A), (ii) the observer's choice of rank-1 idempotent E_{11} (Gap B, step 1), and (iii) the choice of complex structure $u \in S^6$ (Gap B, step 2). Within these conditions, the chain is logically complete.

B. Paper overview

The paper is organized as follows.

Section II (Part A) shows that the C*-algebra nature of the observer forces complexification of the Peirce half-space $V_{1/2}$, upgrading Spin(9) → Spin(10) and $F_4 \to E_6$. This covers links L4–L5 of the chain.

Section III (Part B) constructs the Cl(6) subalgebra from the six directions $W = u^\perp \cap \mathrm{Im}(\mathbb{O})$ defined by the complex structure u. The volume form $\omega_6 = \gamma_1 \cdots \gamma_6$ selects the chiral (left-handed) embedding of the Standard Model fermions via the Pati-Salam [13] intermediate group. This covers link L7.

Section IV presents the synthesis: the single choice of u simultaneously determines both the gauge group (via the F_4 intersection route of Todorov–Drenska [9]) and the chiral representation (via Cl(6)). The three factors $\mathrm{SU}(3)_C$, $\mathrm{SU}(2)_L$, and $\mathrm{U}(1)_Y$ are explicitly matched across both routes. This covers links L8–L9.

Section V presents a detailed gap analysis. We classify the three conditioning inputs (Gap A, Gap B steps 1 and 2) and discuss the open questions of generation structure and the spectral action.

Section VI places the result in the context of the self-modeling research program (Papers 5–6 [1, 2]) and discusses connections to the spectral triple approach [3], Dubois-Violette's earlier work [14], and the broader literature on octonions and exceptional structures in physics [8, 15].

II. PART A: COMPLEXIFICATION FROM C*-OBSERVER

The first half of the derivation shows that the observer's algebraic nature—specifically, the fact that self-modeling forces a C*-algebra structure—automatically complexifies the Peirce half-space of the exceptional Jordan algebra, upgrading the symmetry from Spin(9) to Spin(10) and identifying the 16-dimensional spinor representation that will carry one generation of fermions.

A. The exceptional Jordan algebra $h_3(\mathbb{O})$

The exceptional Jordan algebra $h_3(\mathbb{O})$ (the Albert algebra) consists of 3×3 Hermitian matrices over the octonions $\mathbb{O}$:

$$X = \begin{pmatrix} \alpha & \bar{x}_3 & x_2 \\ x_3 & \beta & \bar{x}_1 \\ \bar{x}_2 & x_1 & \gamma \end{pmatrix}, \qquad \alpha, \beta, \gamma \in \mathbb{R}, \quad x_1, x_2, x_3 \in \mathbb{O}, \tag{1}$$

with the Jordan product

$$A \circ B = \tfrac{1}{2}(AB + BA), \tag{2}$$

where AB denotes ordinary matrix multiplication with octonion entries. The product is well-defined even though $\mathbb{O}$ is non-associative, by the classical result of Albert [7]; the exceptional Jordan algebra was classified by Jordan, von Neumann, and Wigner [5].

Remark 1 (Dimension). The three real diagonal entries contribute 3; the three independent octonion off-diagonal entries contribute $3 \times 8 = 24$. Hence $\dim(h_3(\mathbb{O})) = 3 + 24 = 27$.

Remark 2 (Non-composability and Gap A). Among the formally real Jordan algebras classified by Jordan–von Neumann–Wigner, $h_3(\mathbb{O})$ is the unique *exceptional* algebra: it does not embed into any associative algebra and cannot be realized as a tensor factor of a larger system [8]. We identify it as the "universe algebra" on the following grounds. Under Tegmark's Level IV mathematical universe hypothesis [16] (the program's foundational premise), if an algebra *can* form composites, then composites of it *exist* as consistent mathematical structures. A system whose algebra admits composites is therefore a subsystem of a larger structure, contradicting its identification as the universe. By contrapositive, the universe's algebra must be one that does *not* admit composites: non-composability is required, not merely permitted. This is a separate argument from the self-modeling axioms and is flagged as Gap A.

TABLE I. The 9-link derivation chain from self-modeling to the chiral Standard Model representation. Status: **Proved** = derived in this work or companion papers; *Established* = follows from standard results plus a separate physical argument; `Gap` = an input not derived from self-modeling.

Link	Statement	Source
L1	Self-modeling forces $M_n(\mathbb{C})^{sa}$ (C*-algebra quantum mechanics)	Paper 5 [1]
L2	Non-composability of $h_3(\mathbb{O})$ identifies it as the unique "universe algebra"	JvNW [5] +
L3	Observer selects rank-1 idempotent E_{11}; Peirce decomposition gives $V_1(1) \oplus V_{1/2}(16) \oplus V_0(10)$, $\mathrm{Stab}_{F_4}(E_{11}) = \mathrm{Spin}(9)$	Gap B, step
L4	C*-observer forces complexification: $V_{1/2} \otimes_{\mathbb{R}} \mathbb{C} = S_{10}^{+}$, Spin(9) → Spin(10)	Sec. II
L5	Complexification upgrades $F_4 \to E_6$, $\mathbf{27} \to \mathbf{1} \oplus \mathbf{10} \oplus \mathbf{16}$ under Spin(10)	Sec. II
L6	Observer selects complex structure $u \in S^6 \subset \mathrm{Im}(\mathbb{O})$; $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$	Gap B, step
L7	u defines Cl(6) ⊂ Cl(10); volume form ω_6 selects LEFT embedding: $\mathbf{16} \to (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$	Sec. III
L8	Same u breaks $F_4 \supset [\mathrm{SU}(3)_C \times \mathrm{SU}(3)_J]/\mathbb{Z}_3$; intersection with Spin(9) gives $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$	Sec. IV
L9	Cl(6)/Pati-Salam route gives **same** SM gauge group as F_4 intersection, **plus** chiral representation	Sec. IV

B. Peirce decomposition under E_{11}

The observer selects a rank-1 idempotent

$$e = E_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

Remark 3 (Gap B, step 1). The choice of E_{11} is an *input*, not a consequence of self-modeling. All rank-1 idempotents in $h_3(\mathbb{O})$ are conjugate under F_4 (the orbit is the octonionic projective plane $\mathbb{O}P^2$, dim = 16). Physically, the observer's existence breaks $F_4 \to$ Spin(9); *which* idempotent is selected corresponds to the observer's "viewpoint" and is a genuine symmetry-breaking input.

The Peirce decomposition under e partitions $h_3(\mathbb{O})$ into the eigenspaces of the multiplication operator $L_e(X) = e \circ X$:

$$h_3(\mathbb{O}) \;=\; V_1 \oplus V_{1/2} \oplus V_0, \qquad V_\lambda \;=\; \{X \in h_3(\mathbb{O}) : e \circ X = \lambda X\}. \tag{4}$$

Explicit computation of $L_e(X)$. Writing $E_{11}\, X + X\, E_{11}$ entry by entry and dividing by 2:

$$e \circ X \;=\; \frac{1}{2} \begin{pmatrix} 2\alpha & \bar{x}_3 & x_2 \\ x_3 & 0 & 0 \\ \bar{x}_2 & 0 & 0 \end{pmatrix}. \tag{5}$$

Reading off the eigenvalue conditions:

Proposition 4 (Peirce spaces). *The Peirce decomposition of $h_3(\mathbb{O})$ under E_{11} is:*

$$V_1 = \{\alpha E_{11} : \alpha \in \mathbb{R}\} \cong \mathbb{R}, \qquad \dim(V_1) = 1 \tag{6}$$

$$V_{1/2} = \left\{ \begin{pmatrix} 0 & \bar{x}_3 & x_2 \\ x_3 & 0 & 0 \\ \bar{x}_2 & 0 & 0 \end{pmatrix} : x_2, x_3 \in \mathbb{O} \right\} \cong \mathbb{O}^2, \qquad \dim(V_{1/2}) = 1 \tag{7}$$

$$V_0 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & \bar{x}_1 \\ 0 & x_1 & \gamma \end{pmatrix} : \beta, \gamma \in \mathbb{R},\ x_1 \in \mathbb{O} \right\} \cong h_2(\mathbb{O}), \qquad \dim(V_0) = 1 \tag{8}$$

Dimension check: $1 + 16 + 10 = 27 = \dim(h_3(\mathbb{O}))$. ✓

Stabilizer and Spin(9) representations. The stabilizer of E_{11} in $F_4 = \mathrm{Aut}(h_3(\mathbb{O}))$ (dim $F_4 = 52$) is [8, 15]

$$\mathrm{Stab}_{F_4}(E_{11}) \;=\; \mathrm{Spin}(9), \qquad \dim \mathrm{Spin}(9) = 36 = \tbinom{9}{2}. \tag{9}$$

Coset check: $52 - 36 = 16 = \dim(V_{1/2})$, consistent with $F_4/\mathrm{Spin}(9) \cong \mathbb{O}P^2$.

Under Spin(9) the Peirce spaces carry the representations:

Peirce space	Identification	$\dim_{\mathbb{R}}$	Spin(9) rep
V_1	$\mathbb{R} \cdot E_{11}$	1	trivial $\mathbf{1}$
$V_{1/2}$	$\mathbb{O}^2$	16	spinor $S_9 = \mathbf{16}$
V_0	$h_2(\mathbb{O})$	10	$\mathbf{9} \oplus \mathbf{1}$
Total	$h_3(\mathbb{O})$	27	$\mathbf{1} \oplus \mathbf{16} \oplus \mathbf{9} \oplus \mathbf{1}$

The identification $V_{1/2} = \mathbb{O}^2 = S_9$ (the unique real 16-dimensional irreducible spinor of Spin(9)) follows from $\dim_{\mathbb{R}}(S_9) = 2^{[9/2]} = 2^4 = 16$ and the fact that Spin(9) acts on the tangent space $T_{E_{11}}(\mathbb{O}P^2) \cong \mathbb{O}^2$ via its spinor representation [8, 15]. The space $V_0 = h_2(\mathbb{O})$ decomposes into the traceless part (the 9-dimensional vector representation of $\mathrm{SO}(9) = \mathrm{Spin}(9)/\mathbb{Z}_2$) plus the trace (a trivial $\mathbf{1}$).

C. Complexification from C*-observer nature

The crucial step of Part A is that *complexification is derived, not assumed.* The logical chain has five steps.

Step 1: Self-modeling forces a C*-algebra observer. Paper 5 established that a finite-dimensional system that faithfully self-models must be described by $M_n(\mathbb{C})^{\mathrm{sa}}$—the self-adjoint part of a complex matrix algebra—equipped with the Lüders sequential product. The key steps were: (i) self-modeling forces a sequential product satisfying axioms S1–S7; (ii) the van de Wetering classification yields an EJA state space; (iii) compositionality forces local tomography; (iv) local tomography excludes all non-complex EJA types ($M_n(\mathbb{R})^{\mathrm{sa}}$, $M_n(\mathbb{H})^{\mathrm{sa}}$, spin factors V_n for $n \geq 4$, and $h_3(\mathbb{O})$ itself). The surviving type is $M_n(\mathbb{C})^{\mathrm{sa}}$, promoted to a C*-algebra.

Step 2: C*-algebra implies $\mathbb{C}$ as scalar field. By definition, a C*-algebra is an algebra over $\mathbb{C}$ with an involution and a complete submultiplicative norm satisfying the C*-identity. The observer's scalar field is therefore $\mathbb{C}$, not $\mathbb{R}$.

Step 3: Observer probes $V_{1/2}$ through its own framework. The observer accesses $V_{1/2} = \mathbb{O}^2$ (a real 16-dimensional space) through the Peirce decomposition. Physically, $V_{1/2}$ is the "interaction space" between the observer's slot (V_1) and the rest of the algebra (V_0).

Step 4: Observer's operations are $\mathbb{C}$-linear. Since the observer's scalar field is $\mathbb{C}$ (Step 2), all its algebraic operations—measurement maps, expectation-value functionals, symmetry transformations—are $\mathbb{C}$-linear. There is no mechanism by which the observer can "turn off" its complex structure when probing $V_{1/2}$.

Step 5: Extension of scalars. The observer's effective description of $V_{1/2}$ is therefore

$$V_{1/2}^{\mathbb{C}} \;:=\; V_{1/2} \otimes_{\mathbb{R}} \mathbb{C}\,, \tag{10}$$

the standard extension of scalars from $\mathbb{R}$ to $\mathbb{C}$. Every real basis vector v of $V_{1/2}$ becomes a complex basis vector $v \otimes 1$, so

$$\dim_{\mathbb{C}}\!\big(V_{1/2}^{\mathbb{C}}\big) \;=\; \dim_{\mathbb{R}}(V_{1/2}) \;=\; 16. \tag{11}$$

Remark 5 (Contrast with Boyle [11]). Boyle starts from the complexified algebra $h_3^{\mathbb{C}}(\mathbb{O})$ and studies its structure group E_6. Our approach differs in that we derive the complexification from the observer's C*-nature: the self-modeling requirement (Paper 5) forces the observer to be a C*-algebra system, which in turn forces extension of scalars on every space the observer probes. The end result is the same identification of $V_{1/2}^{\mathbb{C}}$ as the Weyl spinor S_{10}^{+}, but the logical route is different: Boyle assumes complexification, we derive it.

D. Spin(9) to Spin(10) and F_4 to E_6

The complexification of $V_{1/2}$ triggers a representation-theoretic upgrade from Spin(9) to Spin(10).

Proposition 6 (Spinor upgrade)**.** *The complexified Peirce half-space $V_{1/2}^{\mathbb{C}} = \mathbb{O}^2 \otimes_{\mathbb{R}} \mathbb{C}$ carries the Weyl spinor representation S_{10}^{+} of* Spin(10)*:*

$$S_{10}^{+}\big|_{\mathrm{Spin}(9)} \;\cong\; S_9 \otimes_{\mathbb{R}} \mathbb{C} \;=\; S_9^{\mathbb{C}}. \tag{12}$$

Argument. Spin(10) has two inequivalent complex Weyl representations S_{10}^{+} and S_{10}^{-}, each of $\dim_{\mathbb{C}} = 2^{[10/2]-1} = 2^4 = 16$. The standard branching rule for the embedding $\mathrm{Spin}(9) \hookrightarrow \mathrm{Spin}(10)$ (where Spin(9) stabilizes a unit vector in $\mathbb{R}^{10}$) gives $S_{10}^{+}|_{\mathrm{Spin}(9)} \cong S_9^{\mathbb{C}}$: the Weyl spinor of Spin(10), restricted to Spin(9), is the complexification of the real spinor S_9. Equivalently, S_9 is a real form of S_{10}^{+}.

Since $V_{1/2} = S_9$ (Proposition 4) and $V_{1/2}^{\mathbb{C}} = S_9^{\mathbb{C}}$, we conclude $V_{1/2}^{\mathbb{C}} = S_{10}^{+}$. ☐

Dimension check: $\dim_{\mathbb{C}}(S_{10}^{+}) = 16 = \dim_{\mathbb{R}}(S_9) = \dim_{\mathbb{R}}(V_{1/2})$. ✓

Remark 7 (Choice of S_{10}^{+} vs. S_{10}^{-}). Under $\mathrm{Spin}(10) \supset \mathrm{Spin}(9)$, both Weyl representations restrict to the same real representation S_9 (since S_9 has no chirality). The assignment S_{10}^{+} (rather than S_{10}^{-}) follows the Boyle convention [11]. The chirality distinction will be resolved in Section III via the Cl(6) construction.

Symmetry upgrade. Because $V_{1/2}^{\mathbb{C}} = S_{10}^{+}$ carries a representation of Spin(10) (not merely Spin(9)), the symmetry of the observer's description of the Peirce space upgrades:

$$\mathrm{Spin}(9) \;\longrightarrow\; \mathrm{Spin}(10). \tag{13}$$

At the algebra level, the upgrade extends to the full automorphism and structure groups. The structure group of the complexified algebra $h_3^{\mathbb{C}}(\mathbb{O}) = h_3(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$ is [8, 11, 15]

$$F_4 \;\longrightarrow\; E_6, \qquad \dim F_4 = 52 \;\to\; \dim E_6 = 78. \tag{14}$$

Here $F_4 = \mathrm{Aut}(h_3(\mathbb{O}))$ is the automorphism group, and $E_6 = \mathrm{Str}_0(h_3^{\mathbb{C}}(\mathbb{O}))$ is the connected component of the structure group (transformations preserving the cubic norm form up to scale). The stabilizer of E_{11} in E_6 is

$$\mathrm{Stab}_{E_6}(E_{11}) \;=\; \mathrm{Spin}(10) \times \mathrm{U}(1), \qquad \dim = 45 + 1 = 46. \tag{15}$$

Under Spin(10), the 27-dimensional representation decomposes as

$$\mathbf{27} \;\to\; \underbrace{\mathbf{1}}_{V_1^{\mathbb{C}}} \;\oplus\; \underbrace{\mathbf{10}}_{V_0^{\mathbb{C}}} \;\oplus\; \underbrace{\mathbf{16}}_{V_{1/2}^{\mathbb{C}} = S_{10}^{+}}\,, \tag{16}$$

where $V_1^{\mathbb{C}} \cong \mathbb{C}$ (trivial), $V_0^{\mathbb{C}} \cong \mathbf{10}$ (vector of Spin(10), verified by the branching $\mathbf{10}|_{\mathrm{Spin}(9)} = \mathbf{9} \oplus \mathbf{1}$), and $V_{1/2}^{\mathbb{C}} = S_{10}^{+}$ (Weyl spinor).

Dimension check: $1 + 10 + 16 = 27$. ✓

Remark 8 (Cross-check with Boyle). Boyle [11] arrives at the same decomposition $\mathbf{27} = \mathbf{1} \oplus \mathbf{10} \oplus \mathbf{16}$ under $\mathrm{Spin}(10) \subset E_6$ by analyzing the complexified algebra directly. The results are consistent; our contribution is the derivation of the complexification step from C*-observer nature rather than taking it as a mathematical starting point.

III. PART B: CL(6) CHIRALITY AND THE STANDARD MODEL

The second half of the derivation shows that the same algebraic structure—the exceptional Jordan algebra probed by a C*-observer— yields the Standard Model gauge group with the correct chiral representation. The key mechanism is a Clifford subalgebra Cl(6) induced by a single additional choice: a unit imaginary octonion $u \in S^6$.

A. The octonion splitting $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$

Choose a unit imaginary octonion $u \in S^6 \subset \mathrm{Im}(\mathbb{O})$. Following the Fano convention, we set $u = e_7$.

Remark 9 (Gap B, step 2). The choice of u is a second symmetry-breaking input, independent of the idempotent choice E_{11} (Remark 3). All unit imaginary octonions are conjugate under $G_2 = \mathrm{Aut}(\mathbb{O})$; the orbit is $S^6 = G_2/\mathrm{SU}(3)$, $\dim(S^6) = 14 - 8 = 6$. What selects a particular u is not addressed in this work.

The choice of u embeds $\mathbb{C}$ into $\mathbb{O}$:

$$\mathbb{C} \;=\; \mathrm{span}_{\mathbb{R}}\{1, e_7\} \;\subset\; \mathbb{O}. \tag{17}$$

The orthogonal complement of u in $\mathrm{Im}(\mathbb{O})$ is 6-dimensional:

$$W \;=\; u^{\perp} \cap \mathrm{Im}(\mathbb{O}) \;=\; \mathrm{span}_{\mathbb{R}}\{e_1, e_2, e_3, e_4, e_5, e_6\}, \qquad \dim(W) = 6. \tag{18}$$

The space W inherits a complex structure from left multiplication by u:

$$J\colon W \to W, \qquad J(w) = e_7 \cdot w. \tag{19}$$

To verify $J^2 = -\mathrm{Id}$: for any $w \in W$, the left alternative identity $x \cdot (x \cdot y) = (x \cdot x) \cdot y$ gives

$$J(J(w)) \;=\; e_7 \cdot (e_7 \cdot w) \;=\; (e_7 \cdot e_7) \cdot w \;=\; -w, \tag{20}$$

since $e_7^2 = -1$. The complex structure pairs the six real directions into three complex coordinates: $(e_k,\, e_7 \cdot e_k)$ for $k = 1, 2, 3$. Consequently

$$\mathbb{O} \;=\; \mathbb{C} \oplus \mathbb{C}^3 \qquad \text{(as real vector spaces: } 2+6=8\text{)}, \tag{21}$$

where $W \cong \mathbb{C}^3$ as a complex vector space (with J playing the role of multiplication by i).

Stabilizer. The stabilizer of u in $G_2 = \mathrm{Aut}(\mathbb{O})$ is precisely the group that preserves the complex structure J on $W \cong \mathbb{C}^3$:

$$\mathrm{SU}(3)_C \;:=\; \mathrm{Stab}_{G_2}(u) \;=\; \mathrm{Stab}_{G_2}(e_7), \qquad \dim(\mathrm{SU}(3)_C) = 8. \tag{22}$$

This $\mathrm{SU}(3)_C$ acts on $W \cong \mathbb{C}^3$ via the defining 3-dimensional representation and will be identified with the color group.

B. Cl(6) inside Cl(10)

The 6 directions of W become generators of a Clifford subalgebra inside Cl(10). Recall from Section II D that Spin(10) acts on S_{10}^{+} via the Clifford algebra Cl(10) with generators $\Gamma_1, \ldots, \Gamma_{10}$ satisfying $\{\Gamma_A, \Gamma_B\} = 2\delta_{AB}$. The 6 generators corresponding to the W directions define

$$\gamma_k := \Gamma_k, \quad k = 1, \ldots, 6, \qquad \{\gamma_i, \gamma_j\} = 2\delta_{ij}, \tag{23}$$

generating the subalgebra

$$\mathrm{Cl}(6) \;=\; \mathrm{Alg}(\gamma_1, \ldots, \gamma_6) \;\subset\; \mathrm{Cl}(10). \tag{24}$$

The remaining 4 generators $\Gamma_7, \Gamma_8, \Gamma_9, \Gamma_{10}$ correspond to the complement: the $u = e_7$ direction plus three additional directions from the V_0 Peirce space (the $h_2(\mathbb{O})$ sector).

The Cl(6) *subalgebra is not introduced ad hoc*: it is induced by the choice of complex structure u, which selects the 6 directions $e_1, \ldots, e_6 \in W = u^{\perp} \cap \mathrm{Im}(\mathbb{O})$.

Remark 10 (Fock space vs. spinor dimensions). The Cl(6) Fock space (from 3 creation operators $a_j^{\dagger}$) is 8-dimensional (2^3 states with particle number $N = 0, 1, 2, 3$). The 16-dimensional S_{10}^{+} is a Cl(10) spinor, not a Cl(6) Fock space. The relationship is: Cl(6) acts on S_{10}^{+} via its embedding in Cl(10), and the Cl(6) chirality operator ω_6 decomposes $S_{10}^{+} = S^{+} \oplus S^{-}$ into two 8-dimensional eigenspaces, each carrying a Cl(6) Fock space structure. The 8 left-handed states (one generation of SM fermions) occupy one eigenspace; the 8 right-handed states occupy the other.

Volume form. The volume element of Cl(6) is

$$\omega_6 \;=\; \gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 \gamma_6. \tag{25}$$

Proposition 11 (Properties of ω_6). (a) $\omega_6^2 = -1$.

(b) *The chirality operator* $i\,\omega_6$ *satisfies* $(i\,\omega_6)^2 = +1$.

(c) $P = \frac{1}{2}(1 - i\,\omega_6)$ *is an idempotent projector with* $\mathrm{Tr}(P) = 16$ *on the 32-dimensional Dirac spinor* Δ_{10} *of* Cl(10).

Proof. (a) To compute $\omega_6^2 = (\gamma_1 \cdots \gamma_6)^2$, we move the second copy of each generator past the first. The number of transpositions is $5+4+3+2+1+0 = 15 = 6 \cdot 5/2$. Each transposition costs (-1) (from $\{\gamma_i, \gamma_j\} = 0$ for $i \neq j$), and each $\gamma_k^2 = 1$, giving

$$\omega_6^2 \;=\; (-1)^{15} \cdot 1 \;=\; -1. \tag{26}$$

Equivalently, the general formula $\omega_n^2 = (-1)^{n(n-1)/2}$ gives $(-1)^{15} = -1$ for $n = 6$.

(b) $(i\,\omega_6)^2 = i^2 \cdot \omega_6^2 = (-1)(-1) = +1$.

(c) Since $(i\,\omega_6)^2 = 1$, the element $P = \frac{1}{2}(1 - i\,\omega_6)$ is idempotent: $P^2 = \frac{1}{4}(1 - 2i\,\omega_6 + 1) = P$. For the trace on the 32-dimensional Dirac spinor $\Delta_{10} = S_{10}^+ \oplus S_{10}^-$: the volume form ω_6 anticommutes with each γ_k (since $\gamma_k\omega_6 = (-1)^5\omega_6\gamma_k = -\omega_6\gamma_k$), so its eigenvalues come in ± pairs, giving $\mathrm{Tr}(\omega_6) = 0$. Hence $\mathrm{Tr}(P) = \frac{1}{2}(32 - 0) = 16$. ☐

The projector P selects a 16-dimensional subspace of Δ_{10} corresponding to one generation of Standard Model fermions [17].

C. Pati-Salam breaking

The volume form ω_6 breaks Spin(10) to the subgroup that commutes with it. The 10 Clifford generators split into:

- **6 internal generators** $\gamma_1, \ldots, \gamma_6$ (the W directions from Cl(6));
- **4 external generators** $\Gamma_7, \Gamma_8, \Gamma_9, \Gamma_{10}$ (the complement).

The $\binom{10}{2} = 45$ bivectors $\Gamma_A\Gamma_B$ $(A < B)$ spanning $\mathfrak{spin}(10)$ split into three classes based on their commutation with ω_6:

(i) Internal–internal: $\gamma_i\gamma_j$ for $i, j \in \{1, \ldots, 6\}$, $i < j$. Each factor anticommutes with ω_6, so $(\gamma_i\gamma_j)\,\omega_6 = \omega_6\,(\gamma_i\gamma_j)$: *these commute with* ω_6. Count: $\binom{6}{2} = 15$. They generate $\mathfrak{spin}(6) \cong \mathfrak{su}(4)$.

(ii) External–external: $\Gamma_A\Gamma_B$ for $A, B \in \{7, \ldots, 10\}$, $A < B$. Each external generator commutes with ω_6 (passing through 6 internal generators incurs $(-1)^6 = +1$), so $(\Gamma_A\Gamma_B)\,\omega_6 = \omega_6\,(\Gamma_A\Gamma_B)$: *these commute with* ω_6. Count: $\binom{4}{2} = 6$. They generate $\mathfrak{spin}(4) \cong \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$.

(iii) Mixed: $\gamma_i\Gamma_A$ for $i \in \{1, \ldots, 6\}$, $A \in \{7, \ldots, 10\}$. The internal factor anticommutes with ω_6 while the external factor commutes, so $(\gamma_i\Gamma_A)\,\omega_6 = -\omega_6\,(\gamma_i\Gamma_A)$: *these anticommute with* ω_6 and are not in the stabilizer. Count: $6 \times 4 = 24$.

Stabilizer dimension check: $15 + 6 = 21$ (stabilizer) + 24 (complement) = $45 = \dim \mathfrak{spin}(10)$. ✓

Proposition 12 (Pati-Salam stabilizer)**.** *The stabilizer of* ω_6 *in* Spin(10) *is the Pati-Salam group:*

$$\mathrm{Stab}_{\mathrm{Spin}(10)}(\omega_6) \;=\; \frac{\mathrm{Spin}(6) \times \mathrm{Spin}(4)}{\mathbb{Z}_2} \;\cong\; \frac{\mathrm{SU}(4) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R}{\mathbb{Z}_2}, \quad \dim = 21. \tag{27}$$

Representation decomposition. Under Pati-Salam, the 16-dimensional Weyl spinor S_{10}^+ (established in Section II D as the complexified Peirce half-space $V_{1/2}^{\mathbb{C}}$) decomposes as

$$\mathbf{16} \;\to\; (\mathbf{4}, \mathbf{2}, \mathbf{1}) \;\oplus\; (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}). \tag{28}$$

Dimension check: $4 \times 2 \times 1 + 4 \times 1 \times 2 = 8 + 8 = 16$. ✓

The physical content of each factor:

- $(\mathbf{4}, \mathbf{2}, \mathbf{1})$: 4 colors (3 quark + 1 lepton) × $\mathrm{SU}(2)_L$ doublet × $\mathrm{SU}(2)_R$ singlet = **left-handed fermions** ((u_L, d_L) in 3 colors + (ν_L, e_L)).
- $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$: $\bar{4}$ colors × $\mathrm{SU}(2)_L$ singlet × $\mathrm{SU}(2)_R$ doublet = **right-handed fermions** ((u_R, d_R) in 3 colors + (ν_R, e_R)).

This is the **LEFT (chiral) embedding**: $\mathrm{SU}(2)_L$ acts *only* on the $(\mathbf{4}, \mathbf{2}, \mathbf{1})$ component, i.e. on left-handed fermions. Right-handed fermions are $\mathrm{SU}(2)_L$ singlets. This is precisely the chirality pattern of the Standard Model.

Remark 13 (Left vs. diagonal embedding [18]). In the diagonal embedding, a single SU(2) would act on *both* chiralities symmetrically, producing a non-chiral theory. The theorem of Sawin (as cited in [18]) establishes that the Spin(10) GUT embedding of the SM is the LEFT embedding, not the diagonal. Our construction reproduces this: the volume form ω_6 separates left and right sectors *before* the Pati-Salam breaking, ensuring $\mathrm{SU}(2)_L$ acts only on the left-handed component.

D. SM gauge group from Pati-Salam

The same complex structure $u = e_7$ that produced Cl(6) further breaks SU(4). The fundamental representation **4** of SU(4) = Spin(6) acts on $\mathbb{C}^4$. The complex structure u distinguishes the lepton direction (associated with u itself) from the three quark/color directions:

$$\mathbb{C}^4 = \mathbb{C}_{\text{lepton}} \oplus \mathbb{C}^3_{\text{color}}. \tag{29}$$

The stabilizer of this decomposition in SU(4) is SU(3) × U(1) ⊂ SU(4), giving the Standard Model gauge group:

$$\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \tag{30}$$

with hypercharge $Y = (B - L) + 2J_3^R$, where $B - L$ comes from $\mathrm{SU}(4) \to \mathrm{SU}(3) \times \mathrm{U}(1)_{B-L}$ and J_3^R from $\mathrm{SU}(2)_R$.

Dimension check: $\dim(\mathrm{SU}(3)_C) + \dim(\mathrm{SU}(2)_L) + \dim(\mathrm{U}(1)_Y) = 8 + 3 + 1 = 12$. ✓

Full breaking chain:

$$\mathrm{Spin}(10) \xrightarrow{\omega_6} \frac{\mathrm{SU}(4) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R}{\mathbb{Z}_2} \xrightarrow{u} \mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(\tag{31}$$

Both breaking steps are driven by the same input: the choice of complex structure $u = e_7$ on $\mathrm{Im}(\mathbb{O})$.

E. Quantum numbers of one generation

Following Furey [10], the Witt decomposition of Cl(6) defines creation and annihilation operators

$$a_j = \tfrac{1}{2}(\gamma_{2j-1}+i\gamma_{2j}), \qquad a_j^\dagger = \tfrac{1}{2}(\gamma_{2j-1}-i\gamma_{2j}), \qquad j = 1,2,3, \tag{32}$$

satisfying the canonical anticommutation relations $\{a_i, a_j^\dagger\} = \delta_{ij}$, $\{a_i, a_j\} = 0$. The particle-number operator $N = \sum_j a_j^\dagger a_j$ has eigenvalues $0, 1, 2, 3$ on the $2^3 = 8$ Fock states of the Cl(6) spinor. The volume form acts as

$$\omega_6 = -i \cdot (-1)^N, \tag{33}$$

so even-N states have eigenvalue $-i$ and odd-N states have eigenvalue $+i$. The projector $P = \frac{1}{2}(1 - i\,\omega_6)$ selects one chirality sector.

The Schwinger boson construction from Cl(4) Witt operators b_1, b_2 gives the Cartan generators of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$: $J_3^L = \frac{1}{2}(m_1 + m_2 - 1)$, $J_3^R = \frac{1}{2}(m_1 - m_2)$, where $m_k = b_k^\dagger b_k$.

The complete set of quantum number operators is:

$$B - L = 1 - \tfrac{2}{3}N, \qquad Y = (B-L) + 2J_3^R, \qquad Q = J_3^L + \tfrac{Y}{2} \tag{34}$$

$$T_3^c = \tfrac{1}{2}(n_1 - n_2), \quad T_8^c = \tfrac{1}{2\sqrt{3}}(n_1 + n_2 - 2n_3). \tag{35}$$

All 16 SM states for one generation, with quantum numbers computed as explicit matrix eigenvalues [10, 17], are:

	Particle	N	J_3^L	J_3^R	$B-L$	Y	Q
$(\mathbf{4}, \mathbf{2}, \mathbf{1})$	u_L (r, g, b)	1	$+\frac{1}{2}$	0	$+\frac{1}{3}$	$+\frac{1}{3}$	$+\frac{2}{3}$
	d_L (r, g, b)	1	$-\frac{1}{2}$	0	$+\frac{1}{3}$	$+\frac{1}{3}$	$-\frac{1}{3}$
	ν_L	3	$+\frac{1}{2}$	0	-1	-1	0
	e_L	3	$-\frac{1}{2}$	0	-1	-1	-1
$(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$	u_R (r, g, b)	1	0	$+\frac{1}{2}$	$+\frac{1}{3}$	$+\frac{4}{3}$	$+\frac{2}{3}$
	d_R (r, g, b)	1	0	$-\frac{1}{2}$	$+\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$
	ν_R	3	0	$+\frac{1}{2}$	-1	0	0
	e_R	3	0	$-\frac{1}{2}$	-1	-2	-1

Consistency checks: 8 $\mathrm{SU}(2)_L$ doublets ($J_3^L = \pm\frac{1}{2}$) + 8 singlets ($J_3^L = 0$); 4 color singlets (leptons) + 12 color triplets (quarks: 3 colors × 4); Q distribution $\{+\frac{2}{3} : 6,\ -\frac{1}{3} : 6,\ 0 : 2,\ -1 : 2\}$ matches the Pati-Salam SM assignment; $Y = (B - L) + 2J_3^R$ and $Q = J_3^L + Y/2$ verified for all 16 states. Particle content: $u_L \times 3$, $d_L \times 3$, ν_L, e_L, $u_R \times 3$, $d_R \times 3$, ν_R, e_R — complete one-generation SM fermion content.

Remark 14 (Right-handed neutrino). The **16** of Spin(10) necessarily includes ν_R, a right-handed neutrino that is a singlet under the full SM gauge group. This is a *prediction* shared by all SO(10)-based approaches. If ν_R is realized in nature, it provides a natural candidate for Dirac neutrino masses. If it is absent (or acquires a large Majorana mass), the see-saw mechanism would apply. Either way, the construction accommodates current neutrino phenomenology.

IV. SYNTHESIS: ONE CHOICE, TWO CONSEQUENCES

The preceding two sections developed the Standard Model gauge group and its chiral representation by two apparently different routes: the F_4 intersection (Part A, via Todorov-Drenska) and the Cl(6)/Pati-Salam chain (Part B, via Furey and Todorov). We now prove that both routes share a *single* algebraic input and that the second route provides a strict upgrade of the first.

A. The F_4 intersection route (Todorov-Drenska)

The choice of $u \in S^6$ splits the octonions $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ (Section III A). This splitting induces a subgroup of $F_4 = \mathrm{Aut}(h_3(\mathbb{O}))$: the elements that preserve the decomposition of each octonion entry into its $\mathbb{C}$ and $\mathbb{C}^3$ parts. The result, due to Todorov and Drenska [9] (see also Yokota [15]), is the maximal-rank subgroup

$$\frac{\mathrm{SU}(3)_C \times \mathrm{SU}(3)_J}{\mathbb{Z}_3} \subset F_4, \qquad \dim = 8 + 8 = 16, \tag{36}$$

where $\mathrm{SU}(3)_C = \mathrm{Stab}_{G_2}(u)$ acts on the $\mathbb{C}^3$ factor of each octonion entry (the color group) and $\mathrm{SU}(3)_J = \mathrm{Aut}(h_3(\mathbb{C}))$ acts on the Jordan algebra structure of the $\mathbb{C}$-valued part (a "Jordan flavor" group). The $\mathbb{Z}_3$ quotient arises because the cube-root-of-unity centers of the two SU(3) factors intersect.

The SM gauge group emerges as the intersection of two stabilizers within F_4:

$$\mathrm{Stab}_{F_4}(E_{11}) \cap \frac{\mathrm{SU}(3)_C \times \mathrm{SU}(3)_J}{\mathbb{Z}_3} = \mathrm{Spin}(9) \cap \frac{\mathrm{SU}(3)_C \times \mathrm{SU}(3)_J}{\mathbb{Z}_3} \tag{37}$$

The intersection works as follows: $\mathrm{SU}(3)_C \subset G_2 \subset \mathrm{Spin}(9)$, so it passes through the intersection intact. The $\mathrm{SU}(3)_J = \mathrm{Aut}(h_3(\mathbb{C}))$ factor, intersected with $\mathrm{Stab}(E_{11})$ (fixing the first diagonal entry), reduces to $\mathrm{U}(2)_J = [\mathrm{SU}(2) \times \mathrm{U}(1)]/\mathbb{Z}_2$.

Dimension: $8 + 3 + 1 = 12 = \dim(\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y)$. ✓

The containment is an equality at the Lie algebra level: the intersection has dimension 12 (verified by explicit computation of the common generators in [9], Sec. 4), matching $\dim(\mathrm{SU}(3)_C \times \mathrm{SU}(2) \times \mathrm{U}(1)) = 8 + 3 + 1 = 12$.

Remark 15 (No chirality from F_4). The F_4 route gives the gauge group but does *not* provide a chiral representation. The 27-dimensional representation of $h_3(\mathbb{O})$ under F_4 is real: there is no natural complex structure on **27** from the F_4 perspective alone, and therefore no left/right distinction.

B. The single-input theorem

Both routes begin with the *same* decomposition of $\mathrm{Im}(\mathbb{O})$:

$$\mathrm{Im}(\mathbb{O}) \;=\; \mathrm{span}\{u\} \oplus W, \qquad W = u^{\perp} \cap \mathrm{Im}(\mathbb{O}), \quad \dim(W) = 6. \tag{38}$$

Theorem 16 (Single input)**.** *Let $u \in S^6 \subset \mathrm{Im}(\mathbb{O})$ be a unit imaginary octonion, and let E_{11} be a rank-1 idempotent in $h_3(\mathbb{O})$. Then the single choice of u determines:*

(i) ***The F_4 breaking.*** *The splitting $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ induces $[\mathrm{SU}(3)_C \times \mathrm{SU}(3)_J]/\mathbb{Z}_3 \subset F_4$. Its intersection with* Spin(9) $= \mathrm{Stab}_{F_4}(E_{11})$ *contains* $\mathrm{SU}(3)_C \times \mathrm{SU}(2) \times \mathrm{U}(1)$ (the SM gauge group, dim = 12).

(ii) ***The*** Cl(6) ***construction.*** *The same W provides the 6 generators of* $\mathrm{Cl}(6) \subset \mathrm{Cl}(10)$*, whose volume form ω_6 breaks* $\mathrm{Spin}(10) \to \mathrm{SU}(4) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$*. The same u further breaks* $\mathrm{SU}(4) \to \mathrm{SU}(3)_C \times \mathrm{U}(1)_{B-L}$*, giving the SM gauge group with the LEFT chiral embedding.*

No additional algebraic input beyond u and E_{11} is needed for either route.

The two routes use the identical 6-dimensional subspace W—the same six directions $e_1, \ldots, e_6$ (in the Fano convention with $u = e_7$). Their different outputs arise from different algebraic operations on W:

	Route A (F_4 intersection)	**Route B** (Cl(6)/Pati-Salam)
Input	$u \in S^6$	same $u \in S^6$
From W	$J\colon W \to W$ defines $\mathrm{SU}(3)_C = \mathrm{Stab}_{G_2}(u)$	$\gamma_1, \ldots, \gamma_6$ generate $\mathrm{Cl}(6) \subset \mathrm{Cl}(10)$
Intermediate	$[\mathrm{SU}(3)_C \times \mathrm{SU}(3)_J]/\mathbb{Z}_3 \subset F_4$	ω_6 breaks $\mathrm{Spin}(10) \to$ Pati-Salam
Intersect/break	$\cap$ Spin(9)	same u breaks $\mathrm{SU}(4) \to \mathrm{SU}(3)_C \times \mathrm{U}(1)$
Output	$\mathrm{SU}(3)_C \times \mathrm{SU}(2) \times \mathrm{U}(1)$ (dim = 12)	$\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ (dim = 12)
Chirality	**none** (real **27**)	**LEFT** (complex **16**)

C. Group matching

We verify that the two routes produce the *same* gauge group factor by factor.

$\mathrm{SU}(3)_C$**:** In both routes, $\mathrm{SU}(3)_C$ is the group of transformations of $W \cong \mathbb{C}^3$ preserving the complex structure J defined by u.

- Route A phrases this as $\mathrm{Stab}_{G_2}(u)$: automorphisms of $\mathbb{O}$ fixing u, which automatically preserve $J(w) = u \cdot w$.
- Route B phrases this as $\mathrm{Stab}_{\mathrm{SU}(4)}(u)$: the subgroup of $\mathrm{SU}(4) \cong \mathrm{Spin}(6)$ (acting on the 6 directions of W) that commutes with J. Since J is an orthogonal complex structure on $W \cong \mathbb{R}^6$, its stabilizer in SU(4) is $\mathrm{SU}(3) \times \mathrm{U}(1)$.

Both act on $W \cong \mathbb{C}^3$ via the defining 3-dimensional representation. ∴ **Same** $\mathrm{SU}(3)_C$**.**

SU(2)**:**

- Route A: SU(2) from $\mathrm{U}(2)_J = \mathrm{Stab}_{\mathrm{SU}(3)_J}(E_{11})$, the subgroup of the Jordan flavor $\mathrm{SU}(3)_J$ that preserves E_{11}. It acts on the 2nd and 3rd rows/columns of $h_3(\mathbb{C})$.
- Route B: $\mathrm{SU}(2)_L$ from $\mathrm{Spin}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, the external part of $\mathrm{Stab}_{\mathrm{Spin}(10)}(\omega_6)$. It acts on the 4 external directions (Γ_7–Γ_{10}), which correspond to the V_0 Peirce sector.

Both factors arise from the part of Spin(9) (or Spin(10) after complexification) acting on the "external" directions—the Peirce V_0 sector rather than $V_{1/2}$. The Cl(6)/Pati-Salam route provides the additional information that this is $\mathrm{SU}(2)_L$ with a chiral action ($\mathbf{16} \to (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$), whereas the F_4 route sees only a generic SU(2) without left/right distinction. ∴ **Same** SU(2)**.**

U(1)**:**

- Route A: U(1) from the center of $\mathrm{U}(2)_J = [\mathrm{SU}(2) \times \mathrm{U}(1)]/\mathbb{Z}_2$.
- Route B: $\mathrm{U}(1)_Y$ with hypercharge $Y = (B-L) + 2J_3^R$.

Both are determined by rank counting: the rank-4 Cartan subalgebra of F_4 (or equivalently the maximal torus of Spin(10)) has 4 generators. After removing the 2 Cartan generators of $\mathrm{SU}(3)_C$ and the 1 Cartan generator of SU(2), exactly 1 U(1) remains. ∴ **Same** U(1)**.**

Factor	Route A	Route B	Identification
$\mathrm{SU}(3)_C$	$\mathrm{Stab}_{G_2}(u)$	$\mathrm{Stab}_{\mathrm{SU}(4)}(u)$	Same: preserves $J\colon W \to$ acts on $W \cong \mathbb{C}^3$ via **3**
SU(2)	$\mathrm{U}(2)_J \cap \mathrm{Spin}(9)$	$\mathrm{SU}(2)_L \subset \mathrm{Spin}(4)$	Same: acts on external (directions
U(1)	Center of $\mathrm{U}(2)_J$	$\mathrm{U}(1)_Y$	Same: unique residual Carta
Total	dim = 12	dim = 12	$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$

D. The chiral upgrade theorem

Theorem 17 (One choice, two consequences)**.** *Let $h_3(\mathbb{O})$ be the exceptional Jordan algebra. Assume:*

- (Gap A) *$h_3(\mathbb{O})$ is identified as the universe algebra via non-composability.*
- (Gap B, step 1) *An observer selects a rank-1 idempotent $e = E_{11}$, breaking* $F_4 \to \mathrm{Spin}(9)$*.*
- (Gap B, step 2) *A complex structure* $u \in S^6 \subset \mathrm{Im}(\mathbb{O})$ *is chosen.*

Then:
(a) **Gauge group.** *The intersection of* $\mathrm{Spin}(9) = \mathrm{Stab}_{F_4}(E_{11})$ *with the u-preserving subgroup of* F_4 *is the SM gauge group* $\mathrm{SU}(3)_C \times \mathrm{SU}(2) \times \mathrm{U}(1)$ (dim = 12), *verified by explicit computation [9].*
(b) **Chirality.** *The same* u *defines* $\mathrm{Cl}(6) \subset \mathrm{Cl}(10)$ *whose volume form* ω_6 *selects the chiral* (LEFT) *embedding:* $\mathbf{16} \to (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$ *under Pati-Salam, with* $\mathrm{SU}(2)_L$ *on left-handed fermions only.*
(c) **Upgrade.** *The* Cl(6)*/Pati-Salam route* (b) *provides a chiral representation for the* same *gauge algebra that the* F_4 *intersection* (a) *provides without chirality. The chirality is not an additional postulate—it is a consequence of the same algebraic choice u that gives the gauge group.*

The comparison between the two routes is summarized in Table II.

	F_4 intersection	Cl(6)/Pati-Salam
Gauge algebra	$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ (dim = 12)	Same (dim = 12)
Chirality	**None** — F_4 real, **27** real	**LEFT** — ω_6 provides chirality
Representation	Acts on **27** (no chiral decomposition)	Acts on **16** (one generation, definite chirality)
Input	$E_{11} + u$	Same ($E_{11} + u$)
Cross-check value	Independent verification of gauge group	Strictly more informative (same algebra + chirality)

TABLE II. Comparison of the two routes from the single input $u \in S^6$. The Cl(6)/Pati-Salam route provides a strict upgrade: it delivers the same gauge algebra plus chirality. The F_4 route serves as an independent cross-check on the gauge group from a different algebraic framework.

The full chain:

$$\text{self-modeling} \to h_3(\mathbb{O}) \xrightarrow{E_{11}} \mathrm{Spin}(9) \xrightarrow{\text{C*-complexification}} \mathrm{Spin}(10) \xrightarrow{u \in S^6} \mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \text{ with LEFT embedding.} \tag{39}$$

V. GAP ANALYSIS AND OPEN QUESTIONS

The derivation chain of Table I is conditional on three inputs that are not derived from the self-modeling axioms. This section presents a detailed analysis of these gaps: their nature, severity, and what would close them. Honest gap identification is a first-class component of the result, not an afterthought.

A. Conditional structure of the main result

The nine links of Table I fall into three classes according to their logical dependence on the gaps.

Unconditional (proved within this work and companions):

- **L1:** Self-modeling forces $M_n(\mathbb{C})^{\mathrm{sa}}$ [1].
- **L4:** C*-observer forces complexification $V_{1/2} \otimes_{\mathbb{R}} \mathbb{C} = S_{10}^{+}$ (Sec. II).
- **L5:** Complexification upgrades $F_4 \to E_6$, $\mathbf{27} \to \mathbf{1} \oplus \mathbf{10} \oplus \mathbf{16}$ (Sec. II).

These three links stand regardless of how the gaps are resolved.

Conditional on Gap A (L2):

- L2 itself: non-composability identifies $h_3(\mathbb{O})$ as the universe algebra.
- Everything downstream (L3–L9), since it all takes place within $h_3(\mathbb{O})$.

Conditional on Gap B, step 1 (L3):

- L3: Observer selects E_{11}.
- L4–L5: Peirce decomposition requires E_{11}.
- L7–L9: These require Spin(9) and Spin(10), which come from E_{11}.

Conditional on Gap B, step 2 (L6):

- L6: Observer selects $u \in S^6$.
- L7: Cl(6) chirality requires u.
- L8: F_4 intersection requires u.
- L9: Comparison of the two routes requires both L7 and L8.

The strongest unconditional result:

> If the universe algebra is $h_3(\mathbb{O})$ and an observer selects a rank-1 idempotent E_{11} and a complex structure $u \in S^6$, then the C*-nature of the observer forces complexification ($\mathrm{Spin}(9) \to \mathrm{Spin}(10)$), and u simultaneously gives the SM gauge group $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ and its chiral (LEFT) representation.

The three conditioning inputs (Gap A, Gap B steps 1 and 2) are the *only* additional data beyond the self-modeling axioms.

B. Gap register

Table III classifies the five gaps identified in the derivation chain, ordered by their logical role and severity.
Severity justification:

- **B1 (HIGH):** Without the choice of rank-1 idempotent, there is no Peirce decomposition, no Spin(9), and no downstream chain. The entire derivation from L3 onward requires this input.
- **B2 (HIGH):** Without the choice of u, there is no Cl(6), no chirality, and no SM gauge group. The entire synthesis (L7–L9) requires this input.

TABLE III. Gap register for the derivation chain. Severity reflects how the gap affects the chain if left open. “What would close it” indicates the type of argument or result needed, not a claim that closure is imminent.

Gap	Description	Nature	Severity	What would close it
B1	Rank-1 idempotent E_{11} choice	Symmetry breaking: all rank-1 idempotents conjugate under F_4 (orbit = $\mathbb{O}P^2$, dim 16). The observer’s existence breaks $F_4 \to \mathrm{Spin}(9)$, but *which* idempotent is not addressed.	HIGH	A principle selecting which idempotent the observer occupies
B2	Complex structure $u \in S^6$ choice	Symmetry breaking: all u conjugate under G_2 (orbit = $S^6 = G_2/\mathrm{SU}(3)$, dim 6). What selects a particular u is not addressed.	HIGH	A potential on S^6, or a deeper principle selecting u
A	Non-composability → $h_3(\mathbb{O})$	Standard math (JvNW [5] + BGW [6]) + Level IV realism (composable algebra ⇒ composites exist ⇒ not the universe). The L4 step is the program’s foundational premise, not a new assumption, but it is doing real work here.	HIGH	Deriving non-composability from self-modeling axioms directly, without invoking L4 realism
Gen	Why 3 generations	Open question. The “3” in $h_3(\mathbb{O})$ is suggestive, but no mechanism producing 3 copies of the **16** representation has been established.	LOW	Mechanism producing 3 copies of **16** from $h_3(\mathbb{O})$ structure
C	Complexification (L4)	The argument that a C*-observer forces complexification of $V_{1/2}$ is physical reasoning, not a formal proof. The step from “observer’s internal scalar field is $\mathbb{C}$” to “observer’s probe of $V_{1/2}$ extends scalars” needs a precise theorem about measurement maps.	MEDIUM	A theorem: C*-algebra measurem maps on a real Jordan module necessarily extend to $\mathbb{C}$
SA	Spectral action computation	Deferred: the current work establishes the algebraic input (gauge group + chirality) but not the dynamics (GR + SM Lagrangian).	LOW	Computing the GR + SM Lagran from an $h_3(\mathbb{O})$-based spectral trip

- **A (HIGH):** The identification of $h_3(\mathbb{O})$ as the universe algebra rests on the JvNW classification (standard mathematics) combined with Level IV mathematical realism: if an algebra can form composites, then composites of it exist, making it a subsystem rather than the universe. By contrapositive, the universe’s algebra must be non-composable. L4 realism is the program’s foundational premise (not a new assumption), but it is doing substantive work here. Without it, non-composability would be merely *permitted*, not *required*.

- **C (MEDIUM):** The complexification argument (L4) is the paper’s main claimed novelty. It is labeled “Argued” in the chain table, reflecting its status as a physical argument rather than a formal proof. The key step—that a C*-algebra observer probing a real vector space necessarily complexifies it—is plausible but requires a precise theorem about how C*-measurement maps interact with real target spaces.

- **Gen (LOW for this paper):** The present work addresses one generation of fermions. The question of why there are exactly three generations is explicitly out of scope, though it is a crucial open question for the program.

- **SA (LOW for this paper):** Computing the spectral action (the GR + SM Lagrangian) from the algebraic data established here is a separate research direction, deferred to future work.

C. Structural independence of the gaps

Gaps B1 and B2 are *structurally independent*: resolving one does not constrain the other.

B1 does not constrain B2. The stabilizer $\mathrm{Spin}(9) = \mathrm{Stab}_{F_4}(E_{11})$ contains $G_2 = \mathrm{Aut}(\mathbb{O})$, which acts transitively on S^6. Therefore, after fixing E_{11}, all complex structures $u \in S^6$ remain equivalent—fixing the idempotent does not select a preferred u.

B2 does not constrain B1. The u-preserving subgroup $[\mathrm{SU}(3)_C \times \mathrm{SU}(3)_J]/\mathbb{Z}_3$ acts transitively on the rank-1 idempotents of $h_3(\mathbb{C}) \subset h_3(\mathbb{O})$, but not on all rank-1 idempotents of $h_3(\mathbb{O})$. Fixing u restricts the orbit of idempotents from $\mathbb{O}P^2$ (dim 16) to a smaller manifold, but does not determine a unique E_{11}.

Gap A is logically upstream. Without the identification of $h_3(\mathbb{O})$ as the universe algebra, neither the rank-1 idempotent E_{11} nor the complex structure u is defined. Gap A is a prerequisite for Gaps B1 and B2; closing A would not close B1 or B2, but opening A would render B1 and B2 moot.

Gen and SA are downstream/separate. The generation question (why 3 families) and the spectral action computation do not affect the gauge group + chirality result established in this paper. They represent directions for extending the result, not challenges to its validity within its stated scope.

VI. DISCUSSION

A. The research arc: Papers 5–6–7

This paper completes a trilogy deriving the three pillars of fundamental physics from the self-modeling principle.

Paper 5 [1] — Quantum mechanics from self-modeling. The requirement that a finite-dimensional system contains a faithful model of itself, updated through its boundary, forces quantum mechanics: the state space is isomorphic to $M_n(\mathbb{C})^{sa}$ (the self-adjoint part of a matrix C*-algebra). The result is unconditional: the only input is the self-modeling axiom itself.

Paper 6 [2] — General relativity from self-modeling. Self-modeling locality on a lattice—the condition that the self-modeling update acts through boundaries—forces area-law entanglement entropy. Combined with Jacobson's thermodynamic argument [19], this yields Einstein's field equations $G_{ab} + \Lambda g_{ab} = 8\pi G\, T_{ab}$ as the leading-order effective description at long wavelengths. The additional inputs are the lattice topology, the antiferromagnetic coupling sign, and the Wilsonian continuum limit.

Paper 7 (this work) — Standard Model chirality from $h_3(\mathbb{O})$. A C*-observer probing the exceptional Jordan algebra $h_3(\mathbb{O})$ obtains the SM gauge group $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ with the LEFT chiral representation, from a single algebraic choice: a complex structure $u \in S^6 \subset \mathrm{Im}(\mathbb{O})$. The additional inputs beyond self-modeling are the identification of $h_3(\mathbb{O})$ as the universe algebra (Gap A) and two symmetry-breaking choices (Gaps B1 and B2).

The trilogy thus covers:

Paper	Target	Key result	Additional inputs
5	QM	$M_n(\mathbb{C})^{sa}$ forced	None
6	GR	$G_{ab} + \Lambda g_{ab} = 8\pi G\, T_{ab}$	Lattice, coupling sign
7	SM	$\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ + LEFT chirality	$h_3(\mathbb{O})$, E_{11}, u

Each successive paper requires additional inputs beyond the self-modeling axioms: Paper 5 requires none; Paper 6 requires a lattice topology and coupling sign; Paper 7 additionally requires the $h_3(\mathbb{O})$ identification and two symmetry-breaking choices. The increasing number of inputs reflects the increasing specificity of the target: quantum mechanics is generic to all self-modeling systems, general relativity requires spatial structure, and the specific SM gauge group requires exceptional algebraic structure.

B. Relationship to prior work

Several groups have explored the connection between exceptional algebraic structures and the Standard Model. We compare our approach with four lines of work.

Todorov and Drenska [9, 17]. Todorov and Drenska showed that the F_4 intersection—the subgroup of F_4 that preserves the decomposition $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ induced by a complex structure, intersected with $\mathrm{Spin}(9) = \mathrm{Stab}_{F_4}(E_{11})$—contains the SM gauge group $\mathrm{SU}(3)_C \times \mathrm{SU}(2) \times \mathrm{U}(1)$. *Our contribution:* We reproduce this result (Sec. IV A) and additionally (i) derive the complexification step from the C*-observer nature rather than assuming it, (ii) show that the same algebraic input u that produces the F_4 route simultaneously produces the Cl(6) chirality (Theorem 16), and (iii) embed the result in the self-modeling framework (Paper 5).

Furey [10]. Furey showed that the Clifford algebra Cl(6), via the Witt decomposition, produces automatic chirality for one generation of SM fermions: the volume form ω_6 separates left-handed from right-handed states. *Our contribution:* We reproduce this result (Sec. III) and additionally (i) trace the Cl(6) subalgebra to the choice of complex structure u on $\mathrm{Im}(\mathbb{O})$ (it is induced, not postulated), (ii) show that this same u produces the F_4 route as well (the single-input synthesis of Sec. IV), and (iii) derive the complexification that provides the 16-dimensional complex representation on which Cl(6) acts.

Boyle [11]. Boyle studied the complexified exceptional Jordan algebra $h_3^{\mathbb{C}}(\mathbb{O})$ and its structure group E_6, identifying the Weyl spinor S_{10}^+ and the decomposition $\mathbf{27} = \mathbf{1} \oplus \mathbf{10} \oplus \mathbf{16}$ under Spin(10). *Our contribution:* We arrive at the same decomposition (Sec. II D) but *derive* the complexification from the C*-observer nature (Sec. II C) rather than taking it as a mathematical starting point. The end result is the same identification of $V_{1/2}^{\mathbb{C}} = S_{10}^+$; the logical route differs: Boyle assumes complexification, we derive it from self-modeling.

Connes, Chamseddine, and Marcolli [3]. The spectral triple approach of Connes and collaborators derives the SM Lagrangian from noncommutative geometry with the finite algebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. The choice of this direct-sum algebra is an input matched to observation. As noted in the introduction, simple $M_n(\mathbb{C})$ algebras cannot reproduce the SM within this framework—this is the v4.0 structural obstruction [4] that motivates our turn to $h_3(\mathbb{O})$. *Our contribution:* The present work provides the algebraic input—gauge group and chiral representation—from the $h_3(\mathbb{O})$ structure. The spectral action computation (deriving the full GR + SM Lagrangian from an $h_3(\mathbb{O})$-based spectral triple) is identified as Gap SA and deferred to future work. If successful, this would provide a non-perturbative completion of the CCM program with the finite algebra determined by the self-modeling principle rather than chosen by hand.

Dubois-Violette [14]. Dubois-Violette was among the earliest to connect $h_3(\mathbb{O})$ to the Standard Model, studying the algebra's internal symmetries and their relation to gauge groups. *Our contribution:* We provide the self-modeling framework (Paper 5) that motivates $h_3(\mathbb{O})$ as the universe algebra via non-composability, rather than adopting it as a mathematical starting point.

C. Novelty delineation

The individual algebraic components of our derivation are established results. The contribution of this paper lies in three new connections.

1. **Complexification derived from C*-observer nature (new).** The upgrade Spin(9) → Spin(10) and $F_4 \to E_6$ follows from the self-modeling requirement that the observer be a C*-algebra system (Paper 5 [1]), which forces extension of scalars on every space the observer probes. This replaces the assumption of complexification in [11] with a derivation.
2. Cl(6) **chirality and F_4 intersection traced to the same input u (new).** The literature treats the Cl(6)/Furey route and the F_4/Todorov–Drenska route as separate approaches. Theorem 16 proves they share the same algebraic input ($u \in S^6$) and produce the same gauge group, with the Cl(6) route additionally providing chirality (Theorem 17).
3. **Complete chain from self-modeling to chiral SM (new).** No prior work connects the self-modeling axioms (Paper 5) through $h_3(\mathbb{O})$ complexification (Part A) to the chiral SM representation (Part B) in a single derivation chain with explicit gap identification (Table I, Table III).

The established components on which we build include: the JvNW classification of formally real Jordan algebras [5, 7], the structure of $h_3(\mathbb{O})$ and its automorphism group F_4 [8, 15], the Cl(6) → Pati-Salam → SM chain [10, 17], the F_4 intersection route [9], and the $h_3^{\mathbb{C}}(\mathbb{O})/E_6$ analysis [11].

D. Outlook

We identify five concrete directions for extending the present result, each tied to a specific gap.

1. Closing Gap B1: observer selection principle. Is there a principle within the self-modeling framework that selects which rank-1 idempotent the observer occupies? A natural candidate is an extremality condition: the observer occupies the idempotent that maximizes some measure of self-modeling fidelity. Since all rank-1 idempotents are F_4-conjugate, any such principle would need to invoke additional structure (e.g., the presence of matter or interactions) to break the symmetry.

2. Closing Gap B2: dynamics on S^6. Does the spectral action on an $h_3(\mathbb{O})$-based spectral triple produce a potential on S^6 that selects a preferred complex structure? The manifold S^6 is a natural candidate for a moduli space, and the spectral action might generate an effective potential whose minimum determines u. This would connect Gap B2 to Gap SA.

3. Generation structure. The "3" in $h_3(\mathbb{O})$ (the algebra consists of 3×3 matrices) is suggestive: three off-diagonal octonion entries might correspond to three generations. However, no mechanism producing three copies of the **16** representation has been established. Furey [10] proposed a route via the full octonion algebra $\mathbb{O}$ acting on Cl(6); Boyle [11] explored triality. Making this connection rigorous within the self-modeling framework is an open challenge.

4. Spectral action. Computing the full GR + SM Lagrangian from the algebraic data established here—the $h_3(\mathbb{O})$-based spectral triple with $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ gauge group and chiral representation—is the most direct extension. The spectral action principle [3] provides the framework; the challenge is constructing the Dirac operator and evaluating the action for the exceptional algebra, which is non-associative.

5. Connection to other exceptional structures. The exceptional groups G_2, F_4, E_6, E_7, E_8 appear throughout this work. Their role in the exceptional magic square [8] and in higher-dimensional extensions (e.g., $E_8 \times E_8$ in string theory) suggests further structure beyond the SM that the self-modeling framework might illuminate.

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