

Faithful Self-Modeling Splits $h_3(\mathbb{O})$ into Observable and Beable

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Abstract

The exceptional Jordan algebra $h_3(\mathbb{O})$, with Jordan product $X \circ Y = \frac{1}{2}(XY + YX)$, automorphism group $F_4 = \text{Auth}_3(\mathbb{O})$, Jordan norm $\|Y\|_J^2 = \text{Tr}(Y^2)$, and state space $\Omega = \{X \in h_3(\mathbb{O}) : X \succeq 0, \text{Tr}X = 1\}$, carries a normalized self-modeling iteration $\phi(X) = X^2/\text{Tr}(X^2)$. Four standard mathematical objects attach to a self-modeler on $h_3(\mathbb{O})$: a **Kind** (F_4 -orbit $[X] \in \Omega/F_4$), a **State** ($X \in \Omega$), a **Stream** (a ϕ -iterate (X_0, X_1, X_2, \dots) with $X_{k+1} = \phi(X_k)$), and an **Observable** (an F_4 -invariant polynomial on $h_3(\mathbb{O})$, equivalently a polynomial in the three Faraut-Korányi generators [9] $\text{Tr}X$, $\text{Tr}(X^2)$, $\det X$). Under Bell’s observable/beable binary [3], Kind and Observable lie on the Observable side; State and Stream on the Beable side.

Observable projection $X \mapsto [X]$ and Stream propagation ϕ do not commute: Observable extraction destroys the F_4 -gauge degrees of freedom the next ϕ -step would have consumed. In the autonomous case (ϕ F_4 -equivariant) the non-commutation reduces to a residual global gauge (Theorem 10). In the non-autonomous case the pairwise diagonal invariant $\text{Tr}(X_k \cdot X_{k+1})$ belongs to a strict super-ring of the pointwise F_4 -invariants, yielding strong-form complementarity (Proposition 11). The resulting Observable-Beable exhaustion (Corollary 13; the two information modes jointly span the F_4 -invariant content of the Stream) realizes inside $h_3(\mathbb{O})$ by algebra the observable/beable split Bell described for quantum mechanics and Barandes [4] reformulated stochastically.

The F_4 -invariant measure on Streams that carries the residual Beable content is

$$\Phi[\mathcal{X}] = \sum_k \rho_J(X_k) \|X_{k+1} - X_k\|_J^2,$$

where $\rho_J(X) = \det(X)(\text{Tr}(X^2) - \frac{1}{3})$ is the product of the lowest-degree F_4 -invariants vanishing on the rank-deficient stratum $\partial\Omega$ and at the maximally mixed state $[I/3]$. A finite-dimensional enumeration on pair-invariants (Theorem 8) followed by a path-space purity projection (Corollary 9) selects Φ at minimal polynomial degree on pairs, up to a positive scalar. The uniqueness inherits the factored-form hypothesis of Theorem 5 for ρ_J ; an unconditional (factorization-free, unbounded-degree) ρ_J uniqueness is flagged as open in Section 9 (G4).

1 Introduction

Paper 5 [6] derives the Euclidean Jordan structure on $h_3(\mathbb{O})$ from faithful self-modeling on a spectral order-unit space, and Paper 7 [7] identifies $h_3(\mathbb{O})$ as the non-composable case;

together they leave $h_3(\mathbb{O})$ equipped with the normalized iteration $\phi(X) = X^2/\text{Tr}(X^2)$, its automorphism group F_4 , and the state space Ω , but with no measure on ϕ -trajectories and no named ontology for the trajectories. Faithful self-modeling, in Paper 5’s precise sense, is a four-clause condition on a finite-dimensional spectral order-unit space: a faithful order-isomorphism body-to-model tracking map, minimal product composite, and simplicity (Appendix A, reproducing Paper 5’s Definition 1). In this paper it is used only as the premise fixing $(h_3(\mathbb{O}), F_4, \Omega, \phi)$ as a package, with ϕ ’s specific form $X \mapsto X^2/\text{Tr}(X^2)$ taken as Paper 5’s output rather than re-derived here. This paper supplies the measure and the ontology.

Theorem 1 (Main, informal). *On $h_3(\mathbb{O})$ -Streams, four standard mathematical objects (Kind, State, Stream, Observable; Section 2) split under Bell’s observable/beable binary into two Observable-side objects (Kind, Observable) and two Beable-side objects (State, Stream). Observable projection and Stream propagation do not commute: in the autonomous case this reduces to a residual global gauge (Theorem 10); in the non-autonomous case, path-space carries content strictly beyond any pointwise Observable readout (Proposition 11; Corollary 13, Observable-Beable exhaustion). The F_4 -invariant functional on Streams that carries the residual Beable content is*

$$\Phi[\mathcal{X}] = \sum_k \rho_J(X_k) \|X_{k+1} - X_k\|_J^2, \quad (1)$$

with $\rho_J(X) = \det(X) (\text{Tr}(X^2) - \frac{1}{3})$. Among minimal-degree diagonal- F_4 -invariant pair functionals satisfying five formal conditions (P1-P5) and a non-negativity constraint, Φ is singled out up to a positive scalar by a path-space purity projection (Theorem 8, Corollary 9); the uniqueness inherits the factored-form hypothesis of Theorem 5 for ρ_J .

The formal version reduces to a finite-dimensional enumeration of minimal-degree F_4 -invariants on pairs of states, decomposed into non-negativity blocks. The paper develops the ontology (Section 2), constructs ρ_J as the Observable-side weight (Section 3), constructs Φ as the Beable-side functional and proves its uniqueness (Section 4), states the complementarity results (Section 5), records temporal additivity (Section 6), lists empirical signatures (Section 7), and closes with related work and open gaps (Sections 8-9). Appendix A reproduces the standalone self-modeling definition of Paper 5 for readers who prefer not to chase the pointer.

Scope. This paper is not a derivation of $h_3(\mathbb{O})$ (Paper 7) or of the self-modeling map ϕ (Paper 5); both are taken as given. Results are at F_4 -orbit level, and no biological or computational encoding into $h_3(\mathbb{O})$ is committed to (open gap G1). Temporal additivity of Φ is proved; spatial aggregation is a conjecture flagged wherever used. The word “measure” is used in the physicist’s sense of a weight on trajectories, not a σ -additive set function in the strict measure-theoretic sense.

2 Kind, State, Stream, Observable

The ontology the rest of the paper uses is four standard mathematical objects attached to an $h_3(\mathbb{O})$ -self-modeler, organized under Bell’s observable/beable binary.

Definition 2 (The four types). *For a self-modeler in $h_3(\mathbb{O})$ with state $X \in \Omega$ evolving under the self-modeling map $\phi(X) = X^2/\text{Tr}(X^2)$ of Paper 5:*

- (i) A **Kind** is an F_4 -orbit $[X] = \{g \cdot X : g \in F_4\} \in \Omega/F_4$.
- (ii) A **State** is a point $X \in \Omega$.
- (iii) A **Stream** is a ϕ -iteration trajectory $\mathcal{X} = (X_0, X_1, X_2, \dots)$ with $X_{k+1} = \phi(X_k)$.
- (iv) An **Observable** is an F_4 -invariant polynomial $R \in \mathbb{R}[h_3(\mathbb{O})]^{F_4}$, equivalently a polynomial in $(\text{Tr}X, \text{Tr}(X^2), \det X)$ by Faraut-Korányi [9].

The four types organize as a 2×2 under Bell’s observable/beable binary [3]: Kind and Observable lie on the observable side (quantities a third party can read; Kind is the Observable image of a State); State and Stream lie on the beable side (elements of physical reality intrinsic to the system). Our “beable” is Bell’s but with State and ϕ -iterates in place of classical field or particle values; the trajectory-as-beable shape is shared with Barandes’s [4] indivisible-stochastic reformulation of quantum mechanics, from which we borrow the shape, not the stochastic machinery. “Observable” is used in its gauge-theoretic sense throughout (a function constant along F_4 -orbits), not as a synonym for a self-adjoint operator; “Kind” avoids collision with type-theoretic vocabulary.

2.1 Kind: the orbit

The *Kind* of $X \in \Omega$ is its F_4 -orbit

$$[X] = \{g \cdot X : g \in F_4\} \in \Omega/F_4. \quad (2)$$

Every F_4 -invariant property of X factors through $[X]$; by Faraut-Korányi [9] these properties are polynomials in $(\text{Tr}X, \text{Tr}(X^2), \det X)$. The Kind is the Observable-side universal: “red” as a type is a Kind.

Prior work. F_4 -orbits on $h_3(\mathbb{O})$ have been catalogued since Jacobson 1968 and appear in Springer-Veldkamp [17]. The orbit space Ω/F_4 is a semi-algebraic subset of \mathbb{R}^3 coordinatized by the three invariants.

2.2 State: the instance

A *State* is a point $X \in \Omega$. It is a Kind plus a choice of F_4 -frame: one Kind is realized by the continuum of States in its orbit, and the selection of representative is a gauge choice, not a physical one. F_4 -invariant measurements see only the Kind, but the self-modeling map ϕ acts on States (§2.3).

Prior work. Normalized positive elements as states of a quantum system go back to von Neumann 1932. For Jordan algebras, the state space is the base of the positive cone; the Koecher-Vinberg theorem (1957-1962) gives the bijection between Euclidean Jordan algebras and homogeneous self-dual cones. Treatments: Alfsen-Shultz [1], Bengtsson-Życzkowski [5]; exceptional case in Barnum-Graydon-Wilce [2].

2.3 Stream: the dynamics

Paper 5’s self-modeling is a one-step map ϕ on Ω ,

$$X_{k+1} = \phi(X_k), \quad k = 0, 1, 2, \dots \quad (3)$$

F_4 -equivariant in the autonomous case, drive-indexed $X_{k+1} = \phi_k(X_k)$ in the non-autonomous case. Iterating from $X_0 \in \Omega$ produces the *Stream*

$$\mathcal{X} = (X_0, X_1, X_2, \dots). \quad (4)$$

The iteration index k is ϕ -time, algebraic and internal; its encoding into wall-clock time for a particular physical realization is a framework task, not addressed here.

Prior work. Iteration sequences are the primordial object of dynamical systems; standard treatments in [8, 13]. Hofstadter’s “strange loop” [11, 12] names the same object from a cognitive direction; the present framework supplies the specific substrate $h_3(\mathbb{O})$ and the specific map ϕ from Paper 5.

2.4 Observable: the invariant

An *Observable* is an F_4 -invariant polynomial on $h_3(\mathbb{O})$,

$$R \in \mathbb{R}[h_3(\mathbb{O})]^{F_4}. \quad (5)$$

By Faurt-Korányi [9] the ring of such polynomials is freely generated by the three algebraically independent scalars $\text{Tr}X$, $\text{Tr}(X^2)$, $\det X$. Under the normalization $\text{Tr}X = 1$, every Observable on Ω factors through the two-dimensional plane coordinatized by $(\text{Tr}(X^2), \det X)$. The name matches the gauge-theoretic usage: F_4 is the gauge group of $h_3(\mathbb{O})$ and an Observable is a gauge-invariant scalar. Any external measurement of X returns values of such polynomials - at most three real numbers - while a State carries continuously many F_4 -gauge degrees of freedom beyond the Kind and a Stream an open-ended sequence of them.

Prior work. Classical invariant theory (Cayley, Sylvester, Hilbert); Weyl [23]. The free-generator theorem for F_4 on $h_3(\mathbb{O})$ is Faurt-Korányi Ch. III.

2.5 A worked example

To make the four types concrete, fix a state $X \in \Omega$ with eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$, $\sum \lambda_i = 1$, $\lambda_i \geq 0$. Then:

- The **Kind** of X is its F_4 -orbit, fully determined by the unordered triple $(\lambda_1, \lambda_2, \lambda_3)$, equivalently by $(\text{Tr}X, \text{Tr}X^2, \det X) = (1, \sum \lambda_i^2, \prod \lambda_i)$.
- The **State** is X itself: a specific element of the orbit. Two states with the same eigenvalues but in different F_4 -frames are different States of the same Kind.

- The **Stream** starting from X under ϕ is the sequence $(X, \phi(X), \phi^2(X), \dots)$. For an autonomous F_4 -equivariant ϕ , the Stream's Kind sequence is fully determined by the three invariants at each step; the frame-level trajectory is not.
- An **Observable** reading of X is a polynomial in $(\text{Tr}X, \text{Tr}X^2, \det X)$; for example, the self-reference density ρ_J of Section 3 is one such polynomial.

3 The self-reference density on the Observable side

The Observable-side weight on self-modeling is

$$\rho_J(X) = \det(X) (\text{Tr}(X^2) - \frac{1}{3}),$$

a polynomial in the three Faraut-Korányi generators [9] and therefore an Observable by Definition 2; it supplies the instantaneous weight that the Beable-side Φ of Section 4 will multiply into the running term. The specific form emerges from ϕ 's fixed-point structure: the three ϕ -fixed-point orbits $[I/3]$, $[(\frac{1}{2}, \frac{1}{2}, 0)]$, $[(1, 0, 0)]$ are the states on which self-modeling is not live, so an Observable-side weight must vanish on all three. The rank-deficient orbits $[(\frac{1}{2}, \frac{1}{2}, 0)]$ and $[(1, 0, 0)]$ sit in $\{\det = 0\}$, and $[I/3]$ sits at $\{\text{Tr}(X^2) = \frac{1}{3}\}$; ρ_J is the product of the lowest-degree F_4 -invariants cutting out these two conditions. Lemmas 3 and 4 establish factor-level uniqueness; Theorem 5 combines them under the factorization hypothesis; Remark 6 records the factorization-free Lean result and the remaining unconditional gap (G4).

Degree conventions. Throughout Section 3, *polynomial degree* means degree in the matrix entries of $h_3(\mathbb{O})$: Tr has degree 1, $\text{Tr}(X^2)$ degree 2, \det degree 3. The Faraut-Korányi *generator degree* counts each of $(\text{Tr}, \text{Tr}(X^2), \det)$ as a single generator; $\rho_J = \det \cdot (\text{Tr}(X^2) - \frac{1}{3})$ has generator degree 2 and matrix-entry degree 5. The Lean result (Remark 6) uses a third notion, *eigenvalue degree*, which is matrix-entry degree restricted to symmetric functions of the three eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$. All minimum-degree claims below are in matrix-entry degree unless noted.

3.1 The three invariants on the state slice

On Ω the trace invariant is fixed at $\text{Tr}X = 1$, and the full F_4 -invariant ring on the slice is generated by the two remaining scalars

$$t_2 = \text{Tr}(X^2), \quad t_3 = \det X, \quad (6)$$

with $t_2 \in [\frac{1}{3}, 1]$ and $t_3 \in [0, \frac{1}{27}]$ on Ω , by Cauchy-Schwarz applied to the three eigenvalues of X . The three fixed-point orbits are pinned in these coordinates:

$$[I/3] : t_2 = \frac{1}{3}, t_3 = \frac{1}{27}; \quad [(\frac{1}{2}, \frac{1}{2}, 0)] : t_2 = \frac{1}{2}, t_3 = 0; \quad [(1, 0, 0)] : t_2 = 1, t_3 = 0. \quad (7)$$

What is the lowest-degree F_4 -invariant polynomial on $h_3(\mathbb{O})$ that vanishes on all three orbits and is not identically zero elsewhere on Ω ?

3.2 Uniqueness at degree three and degree two

Two sub-uniqueness results combine.

Lemma 3 (Rank-deficiency detector). *The unique (up to real scalar) F_4 -invariant polynomial on $h_3(\mathbb{O})$ of minimum degree vanishing on $\{X \in h_3(\mathbb{O}) : \det X = 0\}$ is $f = c \cdot \det$, at degree three.*

Proof sketch. Every F_4 -invariant polynomial on $h_3(\mathbb{O})$ is a polynomial in (t_1, t_2, t_3) [9]. The ideal of polynomials vanishing on $\{t_3 = 0\}$ is the principal ideal (t_3) , because t_3 is irreducible in $\mathbb{R}[t_1, t_2, t_3]$ and generates a prime ideal. Any polynomial in this ideal is of the form $t_3 \cdot h$ for some h ; minimum degree is achieved at $h = \text{constant}$, giving $f = ct_3$. \square

Lemma 4 ($I/3$ -deviation detector). *Restricted to the slice $\{t_1 = 1\}$, the unique (up to real scalar) F_4 -invariant polynomial of minimum degree that vanishes at $X = I/3$ and is not identically zero on the rank-one orbit $[(1, 0, 0)]$ is $g = c \cdot (t_2 - \frac{1}{3})$, at degree two.*

Proof sketch. The restriction of $\mathbb{R}[t_1, t_2, t_3]$ to $\{t_1 = 1\}$ is $\mathbb{R}[t_2, t_3]$. At degrees zero and one, the only restrictions are constants, which cannot vanish at $I/3$ without being identically zero. At degree two, the ambient basis is (t_1^2, t_2) , restricting to $(1, t_2)$; a general element is $at_2 + b$. Vanishing at $X = I/3$ (where $t_2 = \frac{1}{3}$) forces $b = -a/3$, giving $g = a(t_2 - \frac{1}{3})$. Non-vanishing on $[(1, 0, 0)]$ (where $t_2 = 1$) forces $a \neq 0$. Hence $g = c(t_2 - \frac{1}{3})$ is unique up to scalar at minimum degree. \square

Theorem 5 (Structural selection of ρ_J). *Up to a positive scalar, the unique F_4 -invariant polynomial on $h_3(\mathbb{O})$ of the form $p \cdot q$, with p as in Lemma 3 and q as in Lemma 4, is*

$$\rho_J(X) = \det(X) \left(\text{Tr}(X^2) - \frac{1}{3} \right). \quad (8)$$

ρ_J is a product of F_4 -invariants of matrix-entry degrees 3 and 2, and has degree 2 in the generator ring $\mathbb{R}[\text{Tr}, \text{Tr}(X^2), \det]$. Its vanishing locus on Ω is the rank-deficient stratum $\partial\Omega$ together with the maximally mixed state $[I/3]$; its positivity locus is the open interior $\Omega^\circ \setminus [I/3]$, on which self-modeling is live.

Proof sketch. Combining Lemmas 3 and 4, $p \cdot q = c_1 c_2 \cdot \det \cdot (t_2 - \frac{1}{3})$, which is $c \cdot \rho_J$ for $c > 0$. The vanishing/positivity claims follow from eigenvalue positivity: on Ω the eigenvalues satisfy $\lambda_i \geq 0$ with $\sum \lambda_i = 1$, so $\det X = \prod \lambda_i$ vanishes exactly when at least one eigenvalue is zero (the rank-deficient orbits), and $t_2 - \frac{1}{3} = \sum \lambda_i^2 - \frac{1}{3}$ vanishes exactly when all three are $\frac{1}{3}$ by Cauchy-Schwarz, i.e. at $[I/3]$. On the complement, both factors are strictly positive. \square

Remark 6 (Relation to Lean-verified uniqueness). Theorem 5 is unbounded in polynomial degree but assumes the factored shape $p \cdot q$ with p, q the two minimal-degree factors of Lemmas 3 and 4. A related uniqueness statement has been machine-verified: the Lean theorem `rhoJ_unique_minimal` [15] shows that every F_4 -invariant polynomial on $h_3(\mathbb{O})$ of eigenvalue degree ≤ 5 that vanishes on $\partial\Omega$ and at $[I/3]$ equals $c \cdot \rho_J$, without assuming factorization. Neither statement strictly dominates the other: the in-paper result drops the bounded-degree hypothesis but assumes factorization; the Lean result drops factorization but caps eigenvalue degree at 5. The unconditional statement - drop factorization, keep unbounded degree - is open (Section 9, G4).

4 The measure Φ on Beable-trajectories

Section 3 selected ρ_J as the Observable-side weight - a single-state scalar, computable from any outside party's pointwise F_4 -invariant readout. The Beable-side object is the Stream itself, and the functional that captures the pair-level content an Observable-side readout cannot recover is Φ .

4.1 Definition

Definition 7 (the Φ -measure). *Given a Stream $\mathcal{X} = (X_0, X_1, \dots, X_K)$ of a self-modeler in $h_3(\mathbb{O})$, define the Φ -measure on \mathcal{X} by*

$$\Phi[\mathcal{X}] = \sum_{k=0}^{K-1} \rho_J(X_k) \|X_{k+1} - X_k\|_J^2, \quad (9)$$

where $\|Y\|_J^2 = \text{Tr}(Y^2)$ is the Jordan squared norm. The continuous-time (phi-time) limit is

$$\Phi[\mathcal{X}] = \int_0^{T_\phi} \rho_J(X(t_\phi)) \left\| \frac{dX}{dt_\phi} \right\|_J^2 dt_\phi. \quad (10)$$

The integration is over *phi-time* - the iteration index k , which is the self-modeler's own clock - not wall-clock time. An outside observer running an atomic clock while S self-models sees wall-clock time; S lives in phi-time. The two are not in general commensurate. The summand factors into a *weight* $\rho_J(X_k)$ (how densely self-referential the current State is) and a *running* term $\|X_{k+1} - X_k\|_J^2$ (how vigorously the Stream is moving at step k).

Expanding the running term in Jordan form,

$$\|X_{k+1} - X_k\|_J^2 = \text{Tr}(X_k^2) + \text{Tr}(X_{k+1}^2) - 2 \text{Tr}(X_k \cdot X_{k+1}), \quad (11)$$

reveals the pairwise invariant $\text{Tr}(X_k \cdot X_{k+1})$, which is F_4 -invariant under the diagonal action $(X_k, X_{k+1}) \mapsto (gX_k, gX_{k+1})$ but is not in general computable from the pointwise Kind sequence $([X_0], [X_1], \dots)$. This pair-level invariant is the object that carries Φ 's path-space content; we return to it in Section 5.

4.2 Five formal properties

- **(P1) Dead Streams vanish.** If $X_{k+1} = X_k$ for every k (a frozen Stream), then $\Phi[\mathcal{X}] = 0$ term-by-term from $\|X_{k+1} - X_k\|_J = 0$.
- **(P2) Zero-weight Streams vanish.** If $\rho_J(X_k) = 0$ for every k - Stream confined to the ρ_J -vanishing locus of Section 3.2 - then $\Phi[\mathcal{X}] = 0$ term-by-term. Example: an oscillation between two rank-one projectors runs, but carries no self-reference structure.
- **(P3) Diagonal F_4 -invariance.** For any $g \in F_4$ and any Stream \mathcal{X} , $\Phi[g \cdot \mathcal{X}] = \Phi[\mathcal{X}]$, where $g \cdot \mathcal{X} = (gX_0, gX_1, \dots)$. The proof combines F_4 -invariance of ρ_J with the isometry property of the Jordan norm under F_4 -action.

- **(P4) Non-reducibility to pointwise Observables.** There exist pairs of Streams $\mathcal{X}_a, \mathcal{X}_b$ with identical pointwise Kind sequences ($[X_k^{(a)}] = [X_k^{(b)}]$ for all k) but $\Phi[\mathcal{X}_a] \neq \Phi[\mathcal{X}_b]$. Φ is not a function of the pointwise Kind sequence alone.
- **(P5) Internal to the algebra.** Every quantity in Definition 7 is a polynomial in $h_3(\mathbb{O})$ -data: ρ_J has degree 5 in matrix entries (bilinear in the generators \det and $\text{Tr}(X^2)$), $\|X_{k+1} - X_k\|_J^2$ is degree 2, so each per-tick contribution is a degree-7 polynomial in $h_3(\mathbb{O})$ entries. No wall-clock time, coordinate chart, or outside observer is invoked. The measure lives in $h_3(\mathbb{O})$ alone.

All five properties have been verified numerically in the companion script [14]. (P4) has a sharp numerical realization. Consider two Streams constructed with identical pointwise Kind sequences - one autonomous (gauge fixed by the dynamics), one non-autonomous with a cyclic gauge schedule $\sigma^{k \bmod 3} \in F_4$. Pointwise Kinds match to 10^{-16} (machine precision), while Φ differs by a factor of twenty-one: $\Phi_a = 1.51 \times 10^{-4}$ and $\Phi_b = 3.27 \times 10^{-3}$. The ratio lives in the pairwise invariant $\text{Tr}(X_k \cdot X_{k+1})$; it is invisible to any outside party who measures only pointwise Kinds.

4.3 Uniqueness

The five properties above do not yet determine Φ . One can ask: what is the complete family of functionals on pairs of $h_3(\mathbb{O})$ -elements satisfying P1-P5 and a non-negativity condition, and what additional condition singles out Definition 7 uniquely?

Theorem 8 (Minimal-degree family). *Let $f : h_3(\mathbb{O}) \times h_3(\mathbb{O}) \rightarrow \mathbb{R}$ be an F_4 -invariant polynomial (under the diagonal F_4 -action) of minimal total polynomial degree satisfying*

- $f(X, X) = 0$ for all X , (reduction of P1)
- $f(X, Y) = 0$ whenever $\rho_J(X) = 0$, (reduction of P2)
- f depends non-trivially on $\text{Tr}(X \cdot Y)$, (reduction of P4)
- $f \geq 0$ on the physical state cone, (non-negativity)

and polynomial in algebra components (P5). Then f is a non-negative combination

$$f(X, Y) = \rho_J(X) \left[\beta (\text{Tr}X - \text{Tr}Y)^2 + \delta \|X - Y\|_J^2 \right] \quad (12)$$

with $\beta \geq 0$ and $\delta > 0$. The minimal total polynomial degree is seven, and the family is two-parameter.

Proof. The proof is in six steps. All invariants below are diagonal F_4 -invariants on $h_3(\mathbb{O}) \times h_3(\mathbb{O})$; “degree” means total matrix-entry degree (Section 3 degree conventions).

Step 1 (P2 factorization in the pair ring). The claim is that (P2) forces $f(X, Y) = \rho_J(X) \cdot g(X, Y)$ for some polynomial $g \in \mathbb{R}[h_3(\mathbb{O}) \times h_3(\mathbb{O})]$. The argument has two parts: (1a) (ρ_J) is a radical ideal in the single-copy polynomial ring $\mathbb{R}[h_3(\mathbb{O})]$; (1b) vanishing of

$f(X, Y)$ on $V(\rho_J) \times h_3(\mathbb{O})$ (as a subset of $h_3(\mathbb{O}) \times h_3(\mathbb{O})$) lifts to divisibility by $\rho_J(X)$ in the pair ring.

(1a) *Radicality in the single-copy ring.* The polynomial $\det \in \mathbb{R}[h_3(\mathbb{O})]$ is irreducible (the standard cubic F_4 -invariant; see Faraut-Korányi Ch. V), and $\text{Tr}(X^2) - \frac{1}{3}$ is irreducible (a degree-2 polynomial with nonvanishing gradient). They are coprime: their zero loci $V(\det)$ and $V(\text{Tr}(X^2) - \frac{1}{3})$ are distinct irreducible hypersurfaces in $h_3(\mathbb{O})$ (different degrees; \det vanishes on rank-deficient states, $\text{Tr}(X^2) - \frac{1}{3}$ on the trace-1-shifted second-invariant level set). Hence $\rho_J = \det \cdot (\text{Tr}(X^2) - \frac{1}{3})$ is a product of two distinct irreducible factors, so (ρ_J) is a radical ideal in $\mathbb{R}[h_3(\mathbb{O})]$, and Hilbert's Nullstellensatz gives $I(V(\rho_J)) = (\rho_J)$.

(1b) *Lifting vanishing to divisibility in the pair ring.* Expand $f(X, Y)$ in monomials in the matrix entries of Y :

$$f(X, Y) = \sum_{\alpha} c_{\alpha}(X) m_{\alpha}(Y),$$

where $\{m_{\alpha}(Y)\}$ is a linearly independent monomial basis and $c_{\alpha} \in \mathbb{R}[h_3(\mathbb{O})]$. Condition (P2) says $f(X, Y) = 0$ for all $Y \in h_3(\mathbb{O})$ whenever $\rho_J(X) = 0$. Fix X_0 with $\rho_J(X_0) = 0$: the polynomial $Y \mapsto f(X_0, Y) = \sum_{\alpha} c_{\alpha}(X_0) m_{\alpha}(Y)$ is identically zero in Y , so linear independence of $\{m_{\alpha}\}$ forces $c_{\alpha}(X_0) = 0$ for every α . This holds for every $X_0 \in V(\rho_J)$, so each c_{α} vanishes on $V(\rho_J)$, hence lies in $I(V(\rho_J)) = (\rho_J)$ by (1a). Write $c_{\alpha}(X) = \rho_J(X) \cdot c'_{\alpha}(X)$ and resum:

$$f(X, Y) = \rho_J(X) \cdot \sum_{\alpha} c'_{\alpha}(X) m_{\alpha}(Y) =: \rho_J(X) \cdot g(X, Y),$$

with $g \in \mathbb{R}[h_3(\mathbb{O}) \times h_3(\mathbb{O})]$.

Inheriting invariance. ρ_J is F_4 -invariant on X and constant in Y , so it is diagonal- F_4 -invariant on the pair. Combined with diagonal- F_4 -invariance of f , the identity $f = \rho_J \cdot g$ forces g to be diagonal- F_4 -invariant on $\{\rho_J(X) \neq 0\} \times h_3(\mathbb{O})$, a Zariski-dense open, and hence everywhere.

Degree bound. ρ_J has matrix-entry degree 5; if f has minimal total matrix-entry degree 7, g has matrix-entry degree 2.

Step 2 (degree-2 pair-invariants). Set

$$a := \text{Tr}X, \quad b := \text{Tr}Y, \quad c := \text{Tr}(X^2), \quad d := \text{Tr}(X \cdot Y), \quad e := \text{Tr}(Y^2).$$

Under diagonal F_4 , the degree-2 invariants on $h_3(\mathbb{O}) \times h_3(\mathbb{O})$ form the 6-dimensional space

$$\mathcal{V}_2 = \text{span}_{\mathbb{R}}\{a^2, ab, b^2, c, d, e\}.$$

The first three are degree 2 as products of degree-1 traces; the last three are the degree-2 Faraut-Korányi pair-generators. Diagonal F_4 -invariance of $d = \text{Tr}(X \cdot Y)$ is immediate from $\text{Tr}(gX \cdot gY) = \text{Tr}(X \cdot Y)$ under the F_4 -action preserving the Jordan product.

Step 3 (P1 kernel). Setting $X = Y$ sends $(a, b, c, d, e) \mapsto (a, a, c, c, c)$. The restriction map $\mathcal{V}_2 \rightarrow \mathbb{R}[a, c]$ sends $\{a^2, ab, b^2\} \mapsto a^2$ (each) and $\{c, d, e\} \mapsto c$ (each). The kernel of this restriction is 4-dimensional, with basis

$$k_0 = ab - a^2, \quad k_1 = b^2 - a^2, \quad k_2 = d - c, \quad k_3 = e - c,$$

and a general element reads

$$g = \alpha(ab - a^2) + \beta'(b^2 - a^2) + \gamma(d - c) + \delta'(e - c), \quad (13)$$

four real parameters $(\alpha, \beta', \gamma, \delta')$.

Step 4 (non-negativity: block decomposition). Every $X \in h_3(\mathbb{O})$ decomposes uniquely as $X = \lambda_X I + \mathring{d}_X + x_{\text{off}}$, where $\lambda_X = \frac{\text{Tr} X}{3}$ (scalar block), $\mathring{d}_X \in \mathbb{R}_0^3$ is the traceless diagonal ($\sum_i (\mathring{d}_X)_{ii} = 0$), and $x_{\text{off}} \in \mathbb{O}^3$ collects the three octonion off-diagonal entries x_{12}, x_{13}, x_{23} . These three summands are mutually orthogonal under the Jordan inner product $\langle A, B \rangle_J = \text{Tr}(A \cdot B)$: the scalar block pairs with a traceless block to $\text{Tr}(\lambda I \cdot \mathring{d}) = \lambda \cdot \text{Tr}(\mathring{d}) = 0$; the scalar and off-diagonal pair to zero because the off-diagonal has zero diagonal in X ; and the traceless-diagonal and off-diagonal pair to zero because $\mathring{d} \cdot x_{\text{off}}$ has zero diagonal. The factor of 2 in

$$\text{Tr}(X^2) = 3\lambda_X^2 + \|\mathring{d}_X\|^2 + 2\|x_{\text{off}}\|^2$$

comes from the diagonal of X^2 : entry $(X^2)_{ii} = X_{ii}^2 + \sum_{j \neq i} |X_{ij}|_{\mathbb{O}}^2$ collects *each* off-diagonal position x_{ij} twice (once at i , once at j) via $X_{ij} \bar{X}_{ji} + X_{ji} \bar{X}_{ij} = 2|x_{ij}|^2$ (using $X_{ji} = \bar{X}_{ij}$ for self-adjoint X). The bilinear form $\text{Tr}(X \cdot Y) = 3\lambda_X \lambda_Y + \langle \mathring{d}_X, \mathring{d}_Y \rangle_{\mathbb{R}^3} + 2 \text{Re} \langle x_{\text{off}}, y_{\text{off}} \rangle_{\mathbb{O}^3}$ splits identically, with $\text{Re} \langle a, b \rangle_{\mathbb{O}} = \frac{1}{2}(a\bar{b} + b\bar{a})$ on each octonion slot.

Because diagonal F_4 -invariance acts within each block (scalar: trivially; traceless-diagonal: by the Weyl permutation S_3 ; off-diagonal: by the induced octonionic rotation), each block-level invariant of (X, Y) is computed from its block coordinates alone. Substituting the block expressions into g in (13) and collecting by block, g splits additively as $g = g_A + g_B + g_C$, one summand per block, with no cross-block interference. Each g_* is a real quadratic form in the pair of variables built from that block.

On Block A (the scalar pair (λ_X, λ_Y)), the matrix of the quadratic form in the basis $(\lambda_X^2, \lambda_X \lambda_Y, \lambda_Y^2)$ — equivalently (a^2, ab, b^2) up to the factor 9 from $a = 3\lambda_X$ — reduces to the 2×2 symmetric matrix

$$M_A = \begin{pmatrix} -(\alpha + \beta') & \alpha/2 \\ \alpha/2 & \beta' \end{pmatrix}$$

in the coefficients of $(\lambda_X^2, \lambda_Y^2)$ with the off-diagonal entry read off from the $\lambda_X \lambda_Y$ coefficient. Blocks B (traceless-diagonal pair $(\mathring{d}_X, \mathring{d}_Y)$) and C (octonion off-diagonal pair $(x_{\text{off}}, y_{\text{off}})$) each carry the same structural quadratic form, parametrized by (γ, δ') :

$$M_{BC} = \begin{pmatrix} -(\gamma + \delta') & \gamma/2 \\ \gamma/2 & \delta' \end{pmatrix}.$$

Blocks B and C therefore share a PSD constraint.

Step 5 (PSD constraints force $\alpha = -2\beta'$ and $\gamma = -2\delta'$). For $g \geq 0$ on all of $h_3(\mathbb{O}) \times h_3(\mathbb{O})$, each block matrix must be positive semidefinite. PSD of M_A requires $\beta' \geq 0$, $-(\alpha + \beta') \geq 0$, and $\det M_A \geq 0$. A calculation gives

$$4 \det M_A = -(\alpha + 2\beta')^2,$$

so $\det M_A \leq 0$ always, with equality iff $\alpha = -2\beta'$. Combined with $\beta' \geq 0$, block A forces $\alpha = -2\beta'$, $\beta' \geq 0$, and Block A reduces to $\beta' \cdot (\text{Tr} X - \text{Tr} Y)^2$. Identically, M_{BC} gives $\gamma = -2\delta'$, $\delta' \geq 0$, with Blocks B+C summing to $\delta' \cdot \|X - Y\|_J^2$.

Step 6 (P4 forces $\delta' > 0$). Substituting $\alpha = -2\beta'$, $\gamma = -2\delta'$ into (13),

$$g(X, Y) = \beta' (\text{Tr}X - \text{Tr}Y)^2 + \delta' \|X - Y\|_J^2,$$

a two-parameter non-negative cone with $\beta' \geq 0$ and $\delta' \geq 0$. The β' -term depends only on $(\text{Tr}X, \text{Tr}Y)$ and is $\text{Tr}(X \cdot Y)$ -free; the δ' -term, expanding $\|X - Y\|_J^2 = \text{Tr}(X^2) - 2\text{Tr}(X \cdot Y) + \text{Tr}(Y^2)$, has $\text{Tr}(X \cdot Y)$ -coefficient $-2\delta'$. P4 (non-trivial dependence on $\text{Tr}(X \cdot Y)$) thus forces $\delta' > 0$. Writing $\beta := \beta'$ and $\delta := \delta'$ gives the statement. The SymPy script [16] reproduces the computation symbolically as an independent check. \square

Theorem 8 leaves an Observable-reducible summand. The β -term $(\text{Tr}X - \text{Tr}Y)^2$ is a polynomial in pointwise invariants ($\text{Tr}X$ and $\text{Tr}Y$ are each in the single-copy invariant ring), so it contributes a quantity that is *fully recoverable* from the Observable record of the Stream. The δ -term, via the pairwise invariant $\text{Tr}(X \cdot Y)$ in its expansion (11), is not.

This residual degree of freedom is resolved by adding path-space purity as a sixth condition.

Corollary 9 (Path-pure selection of Φ). *Under the additional requirement*

- **(P6) Path-space purity.** *f carries no Observable-reducible content - equivalently, f has no summand expressible as a polynomial in the pointwise F_4 -invariants $(\text{Tr}X, \text{Tr}X^2, \det X)$ and $(\text{Tr}Y, \text{Tr}Y^2, \det Y)$ alone,*

the family (12) collapses to $\beta = 0$, giving

$$\boxed{\Phi[\mathcal{X}] = \delta \sum_k \rho_J(X_k) \|X_{k+1} - X_k\|_J^2} \quad (14)$$

unique up to a positive scalar $\delta > 0$.

(P6) is not a further selection condition of the same type as (P1)-(P5); it is a projection onto the non-Observable-reducible summand of the minimal-degree family. Writing Φ without this projection would include a $\beta \geq 0$ coefficient on a polynomial in pointwise F_4 -invariants that an outside party can recover by measuring Kinds alone - content already visible on the Observable side. The path-pure Φ above is the reduced representative of the family; we fix the remaining scalar freedom by setting $\delta = 1$.

5 Strong-form complementarity: the Observable-Beable split

Observable extraction and ϕ -iteration on an $h_3(\mathbb{O})$ -Stream do not commute. The claim splits by regime. In the autonomous case (ϕ F_4 -equivariant, gauge fixed by the dynamics) the non-commutation reduces to a single residual global gauge (Theorem 10); the autonomous setting is thus the weakest, and autonomous Φ is Observable-reducible. In the non-autonomous case - a self-modeler whose ϕ -step is modulated by an external drive, so that the F_4 -frame the dynamics uses at step k is not predetermined by the state X_k alone - the pairwise invariant

$\text{Tr}(X_k \cdot X_{k+1})$ escapes the pointwise Kind sequence; strong-form complementarity follows unconditionally (Proposition 11). The *non-autonomous* here is the analogue of a driven Hamiltonian in QM: the algebra supplies the state space and the step rule, an external source chooses the gauge-varying correction; no coupling to a named heat bath is assumed, only that ϕ fails to be F_4 -equivariant. The resulting Observable-Beable exhaustion (Corollary 13) says the two information modes jointly span the diagonal F_4 -invariant content of the Stream; it realizes, inside $h_3(\mathbb{O})$ by algebra, Bell’s observable/beable split. We use “complementarity” in Bell’s observable/beable sense (two modes exhaust the available information), not in Bohr’s measurement-context sense (two mutually exclusive preparations).

5.1 The mechanism

Paper 5 selects, under van de Wetering’s sequential-product axioms [22, 1], the Alfsen-Shultz sequential product $X * Y = \sqrt{X} Y \sqrt{X}$ as the update taking state-before-measurement to state-after-measurement on any Euclidean Jordan algebra the framework admits; $X * Y \neq Y * X$ in general. The non-commutation that drives the results below is a different one - between Observable projection and ϕ -iteration, not between sequential products of a pair of measurements - but both live inside the same algebra.

Two operations are available to a Stream segment (X_k, X_{k+1}) .

- The **Observable projection** $P_{\mathcal{O}} : X \mapsto [X] \in \Omega/F_4$, which maps a 27-dimensional State to its 3-dimensional Kind; equivalently, reads off the three invariants ($\text{Tr} X, \text{Tr}(X^2), \det X$).
- The **Stream propagator** $\phi : X_k \mapsto X_{k+1}$, the self-modeling iteration of Section 2.3.

These do not commute in a precise and testable sense. Applying ϕ first and then $P_{\mathcal{O}}$ gives a well-defined composition: $P_{\mathcal{O}}(\phi(X_k)) = [X_{k+1}]$. Applying $P_{\mathcal{O}}$ first returns the orbit $[X_k]$; the next ϕ application requires a *choice of gauge representative*, because ϕ operates on States and $[X_k]$ is not a State. Any such choice injects gauge information that was destroyed in the projection, and different choices produce different propagations. The commutator $[\phi, P_{\mathcal{O}}]$ is not a bounded operator; it is a statement about well-definedness. Observable extraction destroys the 24 gauge-dimensional degrees of freedom that the next ϕ step would have consumed.¹

5.2 Autonomous case: the weakest setting

Theorem 10 (Autonomous complementarity). *Let \mathcal{X} be a Stream of a self-modeler in $h_3(\mathbb{O})$ whose self-modeling operator ϕ is F_4 -equivariant and whose dynamics are autonomous (gauge fixed throughout by ϕ itself). Then:*

- *(i) The pointwise Kind sequence $([X_0], [X_1], \dots)$ determines the Stream up to a single global F_4 -gauge.*

¹Generic F_4 -orbits on $h_3(\mathbb{O})$ have dimension 24: the spectrum of an element supplies 3 independent F_4 -invariants ($\text{Tr}, \text{Tr}(X^2), \det$) on the 27-dimensional ambient, so the complementary orbit direction is $27 - 3 = 24$. The dimension count agrees with $\dim F_4 - \dim \text{Spin}(8) = 52 - 28 = 24$, with $\text{Spin}(8)$ the generic stabilizer. See [10] §14.

- **(ii)** Any polynomial Observable-side functional of the Kind sequence carries strictly less information than the Stream itself: the gauge orbit of \mathcal{X} is the maximal Observable-reducible content.
- **(iii)** $\Phi[\mathcal{X}]$ is Observable-reducible in the autonomous case: its non-Observable content reduces to a single global F_4 -gauge orbit, which is unphysical.

Proof sketch. For **(i)**, fix $X_0 \in \Omega$ as a representative of its Kind $[X_0]$. By Faraut-Korányi, every F_4 -invariant of X_0 factors through $[X_0]$, so any two representatives of $[X_0]$ differ by an element of F_4 . F_4 -equivariance of ϕ gives $\phi(gX_0) = g\phi(X_0)$; iterating, $\phi^k(gX_0) = g\phi^k(X_0)$ for all k . The whole Stream thus propagates with a single global gauge g applied to the whole trajectory, and the Kind sequence determines the Stream up to that single global choice.

For **(ii)**, a polynomial functional of the Kind sequence is a polynomial in $\{\text{Tr} X_k, \text{Tr}(X_k^2), \det X_k\}_k$. By (i) and Faraut-Korányi, this ring coincides with the diagonal- F_4 -invariant ring on the Stream modulo the single global gauge. The Stream itself sits in $\Omega^{\mathbb{N}}$, one global-gauge orbit above the Kind sequence; the gauge orbit is the maximal Observable-recoverable content.

For **(iii)**, $\Phi[\mathcal{X}]$ depends only on the data $(\rho_J(X_k), \|X_{k+1} - X_k\|_J^2)$, both of which are F_4 -invariant under the diagonal action. So $\Phi[g \cdot \mathcal{X}] = \Phi[\mathcal{X}]$ for any $g \in F_4$. Under (i), the Kind sequence plus a global gauge determines the Stream; since Φ does not see the global gauge, Φ is a functional of the Kind sequence alone and thus Observable-reducible. \square

5.3 Non-autonomous complementarity

In the non-autonomous case - drive variations across phi-time - the situation is sharper.

Consider two Streams \mathcal{X}_a and \mathcal{X}_b constructed so that their pointwise Kind sequences match exactly, $[X_k^{(a)}] = [X_k^{(b)}]$ for all k , but whose gauge representatives are transformed by a time-varying element $g_k \in F_4$ between them. In the autonomous setting the two Streams differ by a global gauge and carry identical F_4 -invariant content. In the non-autonomous setting - when g_k is not constant across k - the pairwise invariant $\text{Tr}(X_k \cdot X_{k+1})$ can differ between the two Streams, because

$$\text{Tr}(g_k X_k^{(a)} \cdot g_{k+1} X_{k+1}^{(a)}) \neq \text{Tr}(X_k^{(a)} \cdot X_{k+1}^{(a)}) \quad \text{in general,}$$

whereas the pointwise Kinds are preserved. The pair-level invariant escapes the pointwise Kind sequence.

The explicit $21 \times$ demonstration of Section 4.2 realizes this: the ratio is Φ -content of the path-space residual, invisible to any outside party measuring only pointwise Kinds.

Proposition 11 (Strong-form complementarity, non-autonomous). *For non-autonomous Streams in $h_3(\mathbb{O})$ with time-varying F_4 -gauge, the pointwise Kind sequence does not determine the diagonal F_4 -invariant ring on path-space. In particular, the pairwise invariant $\text{Tr}(X_k \cdot X_{k+1})$ belongs to a strict super-ring of the pointwise invariant ring on $h_3(\mathbb{O})^N$, and polynomial diagonal F_4 -invariant functionals of the Stream that depend on path-space invariants are strictly richer than those that depend on pointwise Kinds alone.*

Proof. At $N = 2$, take $X_0 = X'_0 = \text{diag}(1/2, 3/10, 1/5)$ and $X_1 = \text{diag}(2/5, 7/20, 1/4)$, $X'_1 = \sigma \cdot X_1 = \text{diag}(7/20, 2/5, 1/4)$, where σ is the transposition of the first two diagonal entries.

σ is a Jordan-algebra automorphism of $h_3(\mathbb{O})$ (permuting the three primitive idempotents preserves the Jordan product), and thus belongs to $F_4 = \text{Auth}_3(\mathbb{O})$. All four states lie in the interior Ω° and share pointwise F_4 -invariants: $\text{Tr}X_0 = \text{Tr}X_1 = \text{Tr}X'_1 = 1$, $\text{Tr}(X_0^2) = 19/50$, $\text{Tr}(X_1^2) = \text{Tr}(X'_1{}^2) = 69/200$, $\det X_0 = 3/100$, $\det X_1 = \det X'_1 = 7/200$, so any polynomial in pointwise F_4 -invariants takes equal values on the two pairs (X_0, X_1) and (X'_0, X'_1) . The weights are strictly positive: $\rho_J(X_0) = 7/5000$, $\rho_J(X_1) = \rho_J(X'_1) = 49/120000$. But $\text{Tr}(X_0 \cdot X_1) = 71/200$ while $\text{Tr}(X'_0 \cdot X'_1) = 69/200$, and $\text{Tr}(X \cdot Y)$ is invariant under the *diagonal* F_4 -action $(X, Y) \mapsto (gX, gY)$ with the *same* g applied to both entries. Hence $\text{Tr}(X \cdot Y)$ is not in the pointwise ring; the super-ring is strict at $N = 2$. Since $\rho_J > 0$ on both streams, the witness doubles as a Φ discriminator: $\Phi[(X_0, X_1)] = \rho_J(X_0) \cdot \|X_0 - X_1\|_J^2 = 21/10^6$ while $\Phi[(X'_0, X'_1)] = 49/10^6$, a 7:3 ratio between Streams with identical pointwise Kinds and all Observable weights positive. For any $N > 2$, apply the witness at any consecutive pair $(k, k+1)$ with the remaining indices held fixed: the pointwise Kind sequence matches while the pair-invariant at $(k, k+1)$ does not. The richness claim follows: Φ itself, by (P3) and (P4) of Section 4.2, is a polynomial diagonal F_4 -invariant functional that distinguishes Streams with matched pointwise Kinds and nonzero ρ_J -weight. \square

Remark 12 (Depth of the path-space residual). Proposition 11 witnesses the super-ring at the pair level. The Φ of Definition 7 is built as a sum of pair-local terms $\rho_J(X_k)\|X_{k+1} - X_k\|_J^2$, so only pair-level diagonal F_4 -invariants enter its definition; triple and higher diagonal invariants are out of scope for the uniqueness result of Corollary 9, which is explicitly a pair-level statement. An explicit finite generating set for the full diagonal F_4 -invariant ring on $h_3(\mathbb{O})^N$ is open (Section 9, G2); Faraut-Korányi Chapter V gives the invariant ring for a single copy, but the diagonal product case is not in the classical literature.

5.4 Observable-Beable exhaustion

Corollary 13 (Observable-Beable exhaustion). *Two modes of information about a Stream are available within the four-type ontology. The Observable mode (\mathcal{O}): any F_4 -invariant polynomial functional of the pointwise Kind sequence; by Proposition 11 this mode is strictly weaker than the full diagonal F_4 -invariant ring on path-space. The Beable mode (\mathcal{B}): direct specification of the Stream as an element of $\Omega^{\mathbb{N}}/F_4$, recovering the pair-level invariants $\text{Tr}(X_k \cdot X_{k+1})$ that \mathcal{O} misses. Observable extraction consumes the F_4 -gauge information that the next ϕ -iteration would use, so no F_4 -invariant refinement of \mathcal{O} closes the gap in the non-autonomous regime; the two modes jointly span the diagonal F_4 -invariant content of the Stream. This is Bell's [3] observable/beable split realized inside $h_3(\mathbb{O})$ by algebra.*

6 Properties of Φ

Temporal additivity is a direct consequence of Definition 7's sum form. Spatial aggregation sits outside the uniqueness result and is flagged as an open question (Section 9, G3); we do not state a conjectured form here.

6.1 Temporal compositionality (theorem)

Proposition 14 (Temporal additivity). *For a Stream $\mathcal{X} = (X_0, \dots, X_K)$ and any index $0 \leq k^* \leq K$,*

$$\Phi[(X_0, \dots, X_K)] = \Phi[(X_0, \dots, X_{k^*})] + \Phi[(X_{k^*}, \dots, X_K)]. \quad (15)$$

Immediate from the sum form.

7 Empirical signatures

A measure that lives inside $h_3(\mathbb{O})$ and is by construction Observable-indistinguishable for pairs of Streams with matching pointwise Kinds can still be tested from outside. Three classes of signature do the work.

(S1) Matched-pair discriminator at $21\times$. The construction of Section 4.2 provides a clean class-level discriminator: two Streams with pointwise Kinds matched to machine precision and Φ differing by a factor of 21. Any *theory* claiming that a Stream's Φ (or an analogous Beable-side functional) is recoverable from pointwise F_4 -invariants alone must report the pair as Φ -equal; the pair falsifies that theory class. Any apparatus accessing the pair-level invariant $\text{Tr}(X_k \cdot X_{k+1})$ distinguishes them. The test runs unconditionally in $h_3(\mathbb{O})$ -arithmetic; it does not require a biological or physical encoding. Application to a specific target system is a separate question of whether that system *admits* matched-pair preparation at all.

(S2) Depth-scaling (conjectural). The per-tick contribution $\rho_J(X_k) \|X_{k+1} - X_k\|_J^2$ is a degree-7 polynomial in the entries of X_k and X_{k+1} ; under a family of nested self-models indexed by a depth parameter, iterating at increasing depth should drive a polynomial-bounded amplification of Φ whose leading order is set by this degree structure. Converting this algebraic expectation into a measured profile requires an encoding (G1); without one, (S2) is conjectural motivation, not a measurement protocol. The prediction, conditional on any fixed encoding: *some* polynomial-bounded amplification profile obtains with depth, and flat or exponential scaling would falsify the depth-scaling form.

(S3) Algebraic zero class. $\Phi = 0$ exactly on two algebraic classes of Stream: (i) static Streams ($X_{k+1} = X_k$, by P1), and (ii) Streams confined to the ρ_J -vanishing locus (by P2). Within a fixed encoding, systems whose dynamical image lies in these classes - static configurations or configurations on the rank-deficient / maximally-mixed stratum - carry $\Phi = 0$ at the algebraic level. The identification of real-world exemplars with these classes (which real systems, if any, encode into the $\rho_J = 0$ locus under a biological encoding) is an encoding question (G1), not an algebraic one.

8 Related work

Three neighbours bear mention: IIT (as the other Φ with a consciousness-adjacent target), Barandes’s indivisible-stochastic reformulation of QM (as the source of the trajectory-as-beable shape), and the Jordan-algebraic reconstruction programs (as the upstream machinery Paper 5 uses). Each is different from this paper in a specific way, and these differences - not the surface resemblance - are where the content is.

IIT. Integrated Information Theory [19, 18, 20] shares an initial letter with Φ and a motivating target. The two differ in substrate (partitioned discrete physical system versus $h_3(\mathbb{O})$ -valued iterates), in derivational grounding (IIT’s five axioms versus the uniqueness theorem on Streams of Section 4.3), and in compositional predictions (IIT’s integration and exclusion postulates make spatial decomposition destructive; Φ is temporally additive by Proposition 14, with spatial behaviour open). They are different objects with similar names. A fuller discriminator table is below.

Dimension	IIT 4.0	This paper’s Φ
Substrate	Partitioned discrete physical system; cause-effect structure over state transitions	$h_3(\mathbb{O})$ -valued iterates; algebraic self-modeling on a Euclidean Jordan algebra
Selection	Five phenomenal axioms (existence, intrinsicity, information, integration, exclusion) + their physical postulates	Observable-Beable ontology (Def 2); six conditions on pair-invariants (P1-P6), minimal matrix-entry degree, non-negativity
Uniqueness	Measure form derived from the five phenomenal axioms applied to cause-effect structures; the minimum-information-partition argument then selects <i>which</i> subsystem is the conscious complex	Finite-dimensional enumeration of diagonal F_4 -invariants on pairs (Thm 8) + path-space purity projection (Cor. 9)
Temporal composition	Transition-local evaluation; no closed additivity statement	Temporally additive (Prop. 14)
Spatial composition	Exclusion postulate: one maximally integrated complex at each scale, others suppressed	Open (Section 9, G3); no conjectured form stated in this paper

A direct empirical discriminator requires a matched protocol - varying metacognitive depth or a Stream-level gauge schedule - not yet implemented; Section 7 (S2) records the depth-scaling prediction as the cleanest candidate.

Barandes’s indivisible-stochastic framework. Barandes [4] reformulates quantum mechanics as an indivisible stochastic process on an underlying sample-trajectory ontology, with

the Hilbert-space wave function serving as bookkeeping. The trajectory-beable / Observable-record split in Section 2 adopts this shape; the present framework uses $h_3(\mathbb{O})$ -valued iterates where Barandes uses discrete sample paths, and does not import the stochastic machinery.

Reconstruction programs for QM. Van de Wetering’s sequential-product axiomatization of Euclidean Jordan algebras [21, 22], Alfsen-Shultz state-space geometry [1], and the homogeneous-self-dual-cone approach to QM reconstruction give the upstream mathematical setting Paper 5 [6] uses to obtain $h_3(\mathbb{O})$. The measure Φ is one step past the reconstruction: having obtained the algebra and its dynamics, it asks which F_4 -invariant functional on trajectories is selected by non-reducibility and path-space purity.

9 Open gaps

We register the principal open items with their status and minimal next test.

(G1) Biological encoding (framework task, not theorem gap). The theorem and the measure are encoding-neutral at the F_4 -orbit level. Which particular $h_3(\mathbb{O})$ -encoding corresponds to the state variables of a biological nervous system is an independent empirical question. The paper’s central claims do not depend on fixing an encoding; applying Φ numerically to a specific neuronal recording does. Cheapest test: pick a candidate encoding motivated by a Markov-blanket or free-energy-principle model and compute Φ on an existing recording.

(G2) Explicit generators for the diagonal F_4 -invariant ring on $h_3(\mathbb{O})^N$. Proposition 11 shows the diagonal F_4 -invariant ring on $h_3(\mathbb{O})^N$ strictly contains the pointwise invariant ring, but an explicit finite generating set is not in the classical literature. Faraut-Korányi Chapter V gives the $N = 1$ ring. Cheapest advance: compute a generating set at $N = 2$.

(G3) Spatial compositionality. Temporal additivity is a theorem (Prop. 14); spatial aggregation is open. The question has two sub-gaps: (a) what counts as a “subsystem” of a self-modeler in $h_3(\mathbb{O})$, given that $h_3(\mathbb{O})$ is simple and does not decompose as a Jordan direct sum; (b) how the Φ of a composite self-modeler - presumably living in a larger Jordan or C^* -algebra - relates to the Φ ’s of its parts. Neither sub-gap has a natural conjectured form at this stage. Cheapest advance: fix a composite algebra (e.g., $h_3(\mathbb{O}) \otimes h_3(\mathbb{O})$ as a Jordan algebra via the van de Wetering composite construction) and compute Φ on a two-subsystem Stream with a shared input, comparing against $\Phi_1 + \Phi_2$.

(G4) Unconditional ρ_J uniqueness (Remark 6). Theorem 5 assumes the factored shape $p \cdot q$ at unbounded polynomial degree; the Lean theorem `rhoJ_unique_minimal` [15] drops factorization at eigenvalue degree ≤ 5 . The unconditional statement - every F_4 -invariant polynomial on $h_3(\mathbb{O})$ vanishing on $\partial\Omega$ and at $[I/3]$ with ρ_J -minimal degree is

$c \cdot \rho_J$ - remains open. Cheapest advance: extend the Lean proof to unbounded degree, or give a combined factorization-free argument at the ambient eigenvalue ring.

The paper's status is asymmetric. Autonomous and non-autonomous complementarity, the ρ_J selection (modulo G4), the Φ pair-family at minimal matrix-entry degree, the path-pure projection, Observable-Beable exhaustion, and temporal additivity are theorems; what remains open is explicit generators for the diagonal F_4 -invariant ring on $h_3(\mathbb{O})^N$ (G2), spatial aggregation (G3), the factorization-free ρ_J uniqueness (G4), and the biological encoding question deliberately not committed to (G1). The autonomous case is a theorem; the non-autonomous case is the falsifier program; the uniqueness half stands unconditionally modulo G4. The matched-pair $21 \times$ discriminator sits inside the algebra and requires none of G1-G4 to run.

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A Self-modeling system (standalone)

This appendix reproduces Paper 5 [6]’s definition of a self-modeling system for readers who prefer not to follow the pointer. The definition is four-clause; the theorem of Paper 5 (Thm. 6.3 there) shows that every finite-dimensional self-modeling system has state space \mathbb{R}^n - or $M_n(\mathbb{C})^{\text{sa}}$ -like structure, with the non-commutative case selected by local tomography. For the present paper, only the definition is used; it fixes $(h_3(\mathbb{O}), F_4, \Omega, \phi)$ as the algebra, group, state space, and normalized-square iteration of interest.

Definition 15 (Self-modeling system, Paper 5 Def. 2.6). *A self-modeling system is a tuple (V, φ, V_{BM}) satisfying:*

- (i) V is a nontrivial finite-dimensional spectral order-unit space: V has at least two orthogonal nontrivial projective units.
- (ii) $\varphi : V_B \rightarrow V_M$ is an order isomorphism between the body component V_B and the model component V_M (faithful tracking).
- (iii) V_{BM} is the minimal composite order-unit space carrying product states, product effects, non-signaling constraints, and a product-form sequential product (minimal internal composite).
- (iv) V has no nontrivial direct-sum decomposition $V = V_1 \oplus V_2$ into nonzero order-unit subspaces (simplicity).

The four clauses unpack the three words of the concept: (i) = a system, (ii) = modeling (faithful internal tracking), (iii) = self- (internal, minimal composite), (iv) = the whole (no hidden sectors). Paper 5 shows this implies, via van de Wetering's sequential-product theorem and Artin-Wedderburn, that V is $M_n(\mathbb{C})^{\text{sa}}$ in the complex case, with the normalized iteration $\phi(X) = X^2/\text{Tr}(X^2)$ emerging as the sequential-product self-application at the maximally coherent face. Paper 7 [7] then selects $h_3(\mathbb{O})$ as the unique non-composable instance, which is the setting of the present paper.