The Self-Modeling Basin Is Exceptional Supergravity

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Abstract

The self-modeling framework identifies quantum mechanics with faithful self-modeling: a finite-dimensional system that contains a faithful model of itself necessarily has $M_n(\mathbb{C})^{\mathrm{sa}}$ state spaces. A companion paper identifies the exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$ as the unique self-modeling basin, forced by non-composability. We show that the algebraic data of $\mathfrak{h}_3(\mathbb{O})$ uniquely match the exceptional entry in the Gunaydin-Sierra-Townsend (GST) classification of N=2 Maxwell-Einstein supergravity theories: within the GST classification, the algebraic data of $\mathfrak{h}_3(\mathbb{O})$ uniquely select the exceptional entry, whose bosonic Lagrangian includes the Einstein-Hilbert term $-R/2$. An observer at a rank-1 idempotent induces a Peirce decomposition $27 = 1 \oplus 16 \oplus 10$. The C^*-bottleneck projects the Peirce complement $V_0 = \mathfrak{h}_2(\mathbb{O})$ to $\mathfrak{h}_2(\mathbb{C})_u \cong \mathbb{R}^{3,1}$, yielding a 4-dimensional algebraic Minkowski space with conformal algebra $\mathfrak{so}(4,2)$. The unique F_4-invariant cubic form $\det(X)$ on $\mathfrak{h}_3(\mathbb{O})$ (Springer uniqueness), which also governs the self-modeling density ρ_J, serves as the gravitational prepotential. The algebra $\mathfrak{h}_3(\mathbb{O})$ satisfies all three GST hypotheses (degree 3, formally real, positive-definite trace form), and the GST bijection uniquely identifies the algebraic data with the exceptional N=2 Maxwell-Einstein supergravity in five dimensions. The r-map (dimensional reduction) gives the 4-dimensional theory with scalar manifold $E_{7(-25)}/(E_{6(-78)} \times \mathrm{U}(1))$ and 28 vectors (27 matter plus graviphoton), with all couplings determined by $\det(X)$. The N=2 structure appears only as the *output* of the classification, not as an input. The result follows from two inputs (the self-modeling definition and the identification of $\mathfrak{h}_3(\mathbb{O})$ as the basin) plus the shared assumption that the algebraic spacetime carries field-theoretic dynamics on a smooth manifold.

1 Introduction

A self-modeling system is one that contains a faithful model of itself. A companion paper [1] argues that any finite-dimensional system satisfying this definition necessarily has state spaces of the form $M_n(\mathbb{C})^{\text{sa}}$: self-modeling *is* complex quantum mechanics. The derivation proceeds through a chain of algebraic forced moves: the sequential product structure of the model constrains the state space to be a Euclidean Jordan algebra, and faithfulness of the model then selects the complex matrix algebras.

A second companion paper [2] argues that the exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$ is the unique candidate for the "basin" in which self-modeling occurs. The argument is non-composability: $\mathfrak{h}_3(\mathbb{O})$ is the only simple Euclidean Jordan algebra that cannot be realized as a tensor factor of a larger system [3]. An observer (a C*-subalgebra of the form $M_n(\mathbb{C})^{\text{sa}}$) probing $\mathfrak{h}_3(\mathbb{O})$ via a rank-1 idempotent induces a Peirce decomposition $27 = 1 \oplus 16 \oplus 10$. Paper 7 argues that the 16-dimensional Peirce half-space $V_{1/2}$ carries one generation of Standard Model fermions, with chirality and the gauge group $\text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)_Y$ emerging from the observer's complex structure and the F_4 automorphism group.

The present paper is conditional on the results of Papers 5 and 7; if those identifications hold, the results here follow.

This paper addresses the remaining sector: the 10-dimensional Peirce complement $V_0 = \mathfrak{h}_2(\mathbb{O})$. We show that the algebraic data of $\mathfrak{h}_3(\mathbb{O})$ uniquely match the exceptional entry in the Gunaydin-Sierra-Townsend (GST) classification of N=2 Maxwell-Einstein supergravity theories [4, 5]. The identification proceeds in the same spirit as Paper 5: just as the Jordan-von Neumann-Wigner classification identifies the self-modeling state spaces with complex quantum mechanics, the GST classification identifies the algebraic data of $\mathfrak{h}_3(\mathbb{O})$ with the exceptional N=2 MESGT.

1.1 The problem

Every known route to Einstein's equations from algebraic or information-theoretic starting points requires substantial additional structure. Connes' spectral action [6] assumes a manifold and a Dirac operator. Jacobson's thermodynamic argument [7] assumes local Lorentz invariance and a Rindler horizon structure. Farnsworth's non-associative spectral geometry [8] works within $\mathfrak{h}_3(\mathbb{O})$ but has not been extended to include gravity. All three approaches take a spacetime manifold as given.

Our advance is that the algebraic identity of spacetime is not assumed.

The Peirce complement V_0, projected through the C*-bottleneck to $\mathfrak{h}_2(\mathbb{C}) \cong \mathbb{R}^{3,1}$, provides a 4-dimensional space with Minkowski signature and the correct conformal algebra. We obtain the *dimension*, *signature*, and *symmetry algebra* of spacetime from $\mathfrak{h}_3(\mathbb{O})$. The step from this algebraic Minkowski space to a smooth manifold carrying dynamical fields is a shared assumption with every quantum gravity program; we do not claim to derive it. Given this assumption, the algebraic data of $\mathfrak{h}_3(\mathbb{O})$ satisfy all three hypotheses of the GST classification theorem, and the GST bijection uniquely identifies these data with the bosonic sector of the exceptional N=2 MESGT, including the Einstein-Hilbert term $-R/2$.

1.2 Summary of results

The identification chain is:

(i) $\mathfrak{h}_3(\mathbb{O})$ is the self-modeling basin (Paper 7, non-composability).

(ii) An observer at a rank-1 idempotent E induces the Peirce decomposition $V_1 \oplus V_{1/2} \oplus V_0$ with dimensions $1 + 16 + 10$.

(iii) The C*-bottleneck projects $V_0 = \mathfrak{h}_2(\mathbb{O})$ to $\mathfrak{h}_2(\mathbb{C})_u \cong \mathbb{R}^{3,1}$ with Minkowski signature $(+, -, -, -)$. The Kantor-Koecher-Tits construction gives $\mathrm{KKT}(\mathfrak{h}_2(\mathbb{C})_u) = \mathfrak{so}(4, 2)$, the 4-dimensional conformal algebra.

(iv) The unique F_4-invariant cubic form $\det(X)$ on $\mathfrak{h}_3(\mathbb{O})$ (Springer, 1962) serves as the gravitational prepotential. Its role in ρ_J and in gravity is forced by algebraic uniqueness (double duty corollary).

(v) $\mathfrak{h}_3(\mathbb{O})$ satisfies all three GST hypotheses: degree 3, formally real, positive-definite trace form (Proposition 8).

(vi) The GST bijection uniquely identifies the algebraic data of $\mathfrak{h}_3(\mathbb{O})$ with the exceptional 5-dimensional N=2 MESGT.

(vii) The r-map (dimensional reduction on a circle) gives the 4-dimensional special Kähler theory with scalar manifold $E_{7(-25)}/(E_{6(-78)} \times \mathrm{U}(1))$.

(viii) The N=2 label appears as the classification output: it is not assumed in any algebraic step, but enters through the GST theorem whose codomain is the class of N=2 MESGTs.

The result is the complete bosonic Lagrangian

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\text{bos}} = &-\frac{R}{2} + g_{i\bar{j}}\,\partial_\mu z^i \partial^\mu \bar{z}^{\bar{j}} \\
&+ \text{Im}(\mathcal{N}_{IJ})\, F^I_{\mu\nu} F^{J\mu\nu} + \text{Re}(\mathcal{N}_{IJ})\, F^I_{\mu\nu} {*F}^{J\mu\nu},
\end{aligned} \tag{1}
$$

with 28 vectors (27 matter plus graviphoton), scalar manifold $E_{7(-25)}/(E_{6(-78)} \times \mathrm{U}(1))$, and all couplings determined by $\det(X)$.

1.3 Conventions

We work in natural units ($\hbar = c = 8\pi G_N = 1$). The metric signature is $(+,-,-,-)$. The Jordan product is $X \circ Y = \frac{1}{2}(XY + YX)$. The octonion basis follows the Fano plane convention $e_1 e_2 = e_4$, and the complex structure is $u = e_7$ throughout. The Peirce idempotent is E_{11} (the rank-1 projector onto the first diagonal entry of $\mathfrak{h}_3(\mathbb{O})$). The real form of the structure group is $E_{6(-26)}$ (not $E_{6(-78)}$ or $E_{6(6)}$).

2 The self-modeling basin

This section establishes notation and recalls the results from Papers 5 and 7 that serve as inputs to the gravitational identification.

2.1 Self-modeling and quantum mechanics

Paper 5 [1] argues that a finite-dimensional system carrying a faithful self-model necessarily has state spaces $M_n(\mathbb{C})^{\text{sa}}$. The key steps are: (i) the sequential product structure of the model forces the state space to be a Euclidean Jordan algebra (Jordan-von Neumann-Wigner classification [9]), and (ii) faithfulness of the model selects the complex matrix algebras over the real, quaternionic, and spin-factor alternatives. Observers are therefore C*-subsystems.

2.2 The exceptional Jordan algebra

Paper 7 [2] argues that $\mathfrak{h}_3(\mathbb{O})$ is the unique arena for self-modeling. The argument uses non-composability: every simple Euclidean Jordan algebra except $\mathfrak{h}_3(\mathbb{O})$ can be embedded as a tensor factor in a larger Jordan algebra [3]. For a self-modeling basin, this embedding would extend the system, contradicting the requirement that the model be *of itself*. The exceptional algebra $\mathfrak{h}_3(\mathbb{O})$ has no such embedding and is therefore the unique candidate.

The algebra $\mathfrak{h}_3(\mathbb{O})$ consists of 3×3 Hermitian matrices over the octonions $\mathbb{O}$:

$$X = \begin{pmatrix} \alpha & x_3 & \bar{x}_2 \\ \bar{x}_3 & \beta & x_1 \\ x_2 & \bar{x}_1 & \gamma \end{pmatrix}, \quad \alpha, \beta, \gamma \in \mathbb{R},\ x_1, x_2, x_3 \in \mathbb{O}, \tag{2}$$

with $\dim(\mathfrak{h}_3(\mathbb{O})) = 3 + 3 \times 8 = 27$. The Jordan product is $X \circ Y = \frac{1}{2}(XY + YX)$, which is well-defined and commutative despite the non-associativity of $\mathbb{O}$. The algebra is simple, formally real, and has degree 3 (the generic minimal polynomial has degree 3).

The automorphism group is $\mathrm{Aut}(\mathfrak{h}_3(\mathbb{O})) = F_4$, the 52-dimensional exceptional Lie group. The structure group (the group preserving the cubic norm up to scale) is $E_{6(-26)}$, a real form of E_6 with signature (-26) [10].

2.3 Peirce decomposition

An observer occupying a rank-1 idempotent $E = E_{11}$ (the projector onto the first diagonal entry, with $\alpha = 1$ and all other entries zero) induces the Peirce decomposition of $\mathfrak{h}_3(\mathbb{O})$ into eigenspaces of the multiplication operator $L_E(X) = E \circ X$:

$$\mathfrak{h}_3(\mathbb{O}) = V_1(E) \oplus V_{1/2}(E) \oplus V_0(E), \tag{3}$$

with eigenvalues 1, $\frac{1}{2}$, and 0 respectively, and dimensions:

$$27 = \underbrace{1}_{V_1} \oplus \underbrace{16}_{V_{1/2}} \oplus \underbrace{10}_{V_0}. \tag{4}$$

The Peirce sectors have the following structure:

- $V_1 \cong \mathbb{R}$: the observer's degree of freedom (α).
- $V_{1/2} \cong \mathbb{O}^2$: the 16-dimensional interface, parametrized by $(x_2, x_3) \in \mathbb{O}^2$. Paper 7 proved that $V_{1/2}$ carries one generation of Standard Model fermions under $\mathrm{Spin}(10) \to \mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$.
- $V_0 = \mathfrak{h}_2(\mathbb{O})$: the 10-dimensional Peirce complement, parametrized by (β, x_1, γ). This is the observer's "external world" and is the subject of this paper.

The Peirce multiplication rules are [11]:

$$\begin{aligned} V_{1/2} \circ V_{1/2} &\subset V_1 \oplus V_0, \\ V_0 \circ V_0 &\subset V_0, \\ V_{1/2} \circ V_0 &\subset V_{1/2}. \end{aligned} \tag{5}$$

The rule $V_{1/2} \circ V_{1/2} \subset V_1 \oplus V_0$ is physically significant: measurements of the matter sector ($V_{1/2}$) produce outcomes correlated with both the observer (V_1) and the external world (V_0). The rule $V_0 \circ V_0 \subset V_0$ means V_0 is a Jordan subalgebra.

2.4 Observer independence

Different rank-1 idempotents give different Peirce decompositions, but they are all F_4-conjugate. Freudenthal [10] proved that F_4 acts transitively on the set of rank-1 idempotents in $\mathfrak{h}_3(\mathbb{O})$, with stabilizer Spin(9). The orbit is the octonionic projective plane $\mathbb{OP}^2 = F_4/\mathrm{Spin}(9)$, a 16-dimensional compact manifold. Every observer therefore sees the same algebraic structure, up to an F_4 automorphism: the Peirce dimensions, the multiplication rules, and the cubic norm are all observer-independent.

We have verified this computationally for the idempotents E_{11} and E_{22}: the permutation $P = (1, 0, 2)$ is a Jordan automorphism mapping $E_{11} \mapsto E_{22}$, and the induced Peirce decompositions are isomorphic (both have V_0 of dimension 10 with $\det_2$ signature $(+, -, -, -)$).

3 Spacetime from the Peirce complement

The Peirce complement $V_0 = \mathfrak{h}_2(\mathbb{O})$ is a 10-dimensional Jordan subalgebra. We show that the C*-bottleneck projects it to a 4-dimensional Minkowski space $\mathfrak{h}_2(\mathbb{C})_u \cong \mathbb{R}^{3,1}$, and that the Kantor-Koecher-Tits construction on $\mathfrak{h}_2(\mathbb{C})_u$ gives the 4-dimensional conformal algebra $\mathfrak{so}(4, 2)$.

3.1 The projection π_u

The observer's complex structure $u = e_7 \in S^6 \subset \mathrm{Im}(\mathbb{O})$ determines a complex subalgebra $\mathbb{C}_u = \mathbb{R} \oplus \mathbb{R}u \subset \mathbb{O}$. The projection $\pi_u : \mathfrak{h}_2(\mathbb{O}) \to \mathfrak{h}_2(\mathbb{C})_u$ acts on an element $Y \in \mathfrak{h}_2(\mathbb{O})$ by projecting the off-diagonal octonion entry x_1 onto $\mathbb{C}_u$:

$$\pi_u \begin{pmatrix} \beta & x_1 \\ \bar{x}_1 & \gamma \end{pmatrix} = \begin{pmatrix} \beta & \mathrm{proj}_{\mathbb{C}_u}(x_1) \\ \overline{\mathrm{proj}_{\mathbb{C}_u}(x_1)} & \gamma \end{pmatrix}. \tag{6}$$

Proposition 1. *The projection π_u has the following properties:*

(a) π_u is idempotent, with 4-dimensional image $\mathfrak{h}_2(\mathbb{C})_u$ and 6-dimensional kernel W consisting of elements of $\mathfrak{h}_2(\mathbb{O})$ with zero diagonal entries

and off-diagonal octonion component in $\mathrm{span}\{e_1,\dots,e_6\}$ *(the imaginary octonion units orthogonal to u). That is,* $W = \left\{ \left(\begin{smallmatrix} 0 & w \\ \bar{w} & 0 \end{smallmatrix}\right) : w \in \mathrm{span}\{e_1,\dots,e_6\} \right\}$.

(b) *The Gram matrix of* $\det_2$ *restricted to* $\mathfrak{h}_2(\mathbb{C})_u$ *has eigenvalues* $\{+1,-1,-1,-1\}$*: Minkowski signature* $(1,3)$*.*

(c) V_0 *closure under the Jordan product is a standard consequence of the Peirce multiplication rules; we have verified this computationally as a consistency check (all* 55 *basis pair products have zero* $V_{1/2}$*-leakage).*

(d) π_u *is* not *a Jordan homomorphism. The failure is characterized by*

$$\Delta(A,B) := \pi_u(A\circ B) - \pi_u(A)\circ\pi_u(B) = \langle w_A, w_B\rangle_W \cdot I_2, \qquad (7)$$

where $w_A = \mathrm{proj}_W(x_{1,A})$ *is the* W*-component of* A*. In particular,* Δ *vanishes on* $\mathfrak{h}_2(\mathbb{C})_u$ *(where* $w=0$*) and is nonzero only for elements with components in the internal* W*-directions.*

Remark 2. No step in the derivation chain requires π_u to be a Jordan homomorphism. The GST Lagrangian is defined on the full 27-dimensional $\mathfrak{h}_3(\mathbb{O})$ via d_{IJK}; π_u enters only to identify which 4 dimensions constitute spacetime. The failure Δ of Eq. (7) is confined to cross-terms with internal W-directions and does not affect the Peirce block structure of d_{IJK}.

The 10-dimensional V_0 therefore splits cleanly as

$$V_0 = \underbrace{\mathfrak{h}_2(\mathbb{C})_u}_{\text{4-dim, spacetime}} \oplus \underbrace{W}_{\text{6-dim, internal}}, \qquad (8)$$

where $\mathfrak{h}_2(\mathbb{C})_u$ carries Minkowski signature from $\det_2$ and W is the kernel of π_u. The explicit Minkowski coordinates on $\mathfrak{h}_2(\mathbb{C})_u$ are

$$y^0 = \tfrac{1}{2}(\beta+\gamma), \quad y^3 = \tfrac{1}{2}(\beta-\gamma), \quad y^1 = \mathrm{Re}(x_1), \quad y^2 = \langle x_1, e_7\rangle, \qquad (9)$$

where $x_1 \in \mathbb{O}$ is the off-diagonal octonion entry of $Y \in \mathfrak{h}_2(\mathbb{O})$, and $\det_2 = (y^0)^2 - (y^1)^2 - (y^2)^2 - (y^3)^2$.

3.2 Minkowski structure from $V_{1/2}$ products

The Peirce multiplication rule $V_{1/2}\circ V_{1/2} \subset V_1 \oplus V_0$ provides additional structure. The V_0-component of the $V_{1/2}\times V_{1/2}$ product, projected through π_u, yields four 16×16 Minkowski matrices M_μ satisfying

$$\{M_i, M_j\} = \tfrac{1}{2}\,\delta_{ij}\, I_{16}, \quad i,j \in \{1,2,3\}, \qquad M_0 = \tfrac{1}{2}\, I_{16}. \qquad (10)$$

The rescaled matrices $\gamma_i = 2M_i$ satisfy $\{\gamma_i, \gamma_j\} = 2\delta_{ij} I_{16}$, generating $\mathrm{Cl}(3, 0)$ on $\mathbb{R}^{16}$. This Clifford algebra structure means that fermion bilinears in $V_{1/2}$ naturally generate the spatial rotation algebra of $\mathfrak{h}_2(\mathbb{C})_u$, connecting the matter sector to the spacetime sector at the algebraic level.

3.3 Conformal algebra

The Kantor-Koecher-Tits (KKT) construction [12, 13, 14] builds a Lie algebra from any Jordan algebra J (see [11] for a modern treatment):

$$\mathrm{KKT}(J) = J^- \oplus \mathrm{Str}_0(J) \oplus J^+, \tag{11}$$

where J^+ (translations) and J^- (special conformal transformations) are copies of J as a vector space, and $\mathrm{Str}_0(J)$ is the structure algebra (derivations plus multiplication operators).

Proposition 3. *For the observer's spacetime* $\mathfrak{h}_2(\mathbb{C})_u$*:*

- *(a)* $\mathrm{KKT}(\mathfrak{h}_2(\mathbb{C})_u) = \mathfrak{so}(4, 2)$*, the 15-dimensional 4d conformal algebra.*
- *(b) The Killing form of* $\mathrm{KKT}(\mathfrak{h}_2(\mathbb{C})_u)$ *has signature* $(8, 7)$*.*
- *(c) The derivation algebra* $\mathrm{Der}(\mathfrak{h}_2(\mathbb{C})_u)$ *has dimension 3 and generates the rotation subalgebra* $\mathfrak{so}(3)$*.*
- *(d) The structure algebra* $\mathrm{Str}_0(\mathfrak{h}_2(\mathbb{C})_u)$ *has dimension 7 and contains both rotations and Lorentz boosts.*
- *(e) Every rank-1 idempotent in* $\mathfrak{h}_3(\mathbb{O})$ *gives the same conformal algebra (observer independence, via* F_4 *transitivity).*

The conformal algebra $\mathfrak{so}(4, 2)$ contains the Poincaré algebra as a subalgebra: translations (J^+), Lorentz transformations (within Str_0), and dilatations (the grading element). The emergence of $\mathfrak{so}(4, 2)$ from the Peirce complement is purely algebraic: no manifold, metric, or differential structure is assumed. The KKT construction builds these symmetries from the Jordan product alone.

3.4 Uniqueness of the spacetime

Not every 4-dimensional subspace of $V_0 = \mathfrak{h}_2(\mathbb{O})$ gives a spacetime with the correct conformal algebra.

Proposition 4. *Let $J_4 \subset \mathfrak{h}_2(\mathbb{O})$ be a 4-dimensional unital Jordan subalgebra (containing the identity).*

(a) All 4-dimensional unital Jordan subalgebras of $\mathfrak{h}_2(\mathbb{O})$ (containing the identity) are isomorphic to $\mathrm{JSpin}(3)$ *(a spin factor), parametrized by the Grassmannian* $\mathrm{Gr}(3, 9)$.

(b) $\dim \mathrm{KKT}(\mathrm{JSpin}(n)) = (n+3)(n+2)/2$. *Only $n = 3$ gives* $\dim = 15 = \dim \mathfrak{so}(4, 2)$.

(c) The complex structure $u \in S^6$ uniquely selects $\mathfrak{h}_2(\mathbb{C})_u$ as the π_u-image. Different choices of u are related by G_2 automorphisms of $\mathbb{O}$ and give isomorphic spacetimes.

The choice of complex structure u is therefore observer-independent: no physical content depends on which $u \in S^6$ is selected, since G_2 transitivity gives isomorphic Peirce decompositions for every choice. The full orbit is $G_2/\mathrm{SU}(3) \cong S^6$, and G_2 acts as an internal symmetry relating equivalent descriptions.

3.5 Stabilizer structure

The choice of complex structure $u = e_7$ selects a $3+6$ split of the 9 traceless directions of the spin factor $\mathfrak{h}_2(\mathbb{O})$: 3 lie in the traceless subspace of $\mathfrak{h}_2(\mathbb{C})_u$ (the spatial directions y^1, y^2, y^3) and 6 span the complement W (internal). This split determines the internal symmetries visible to the observer.

Proposition 5. *The subgroup of* $\mathrm{Spin}(9)$ *preserving the* $3+6$ *split induced by $u = e_7$ has Lie algebra $\mathfrak{so}(3) \oplus \mathfrak{so}(6)$, dimension 18:*

(a) $\mathfrak{so}(3)$ *acts on the spacetime indices* $\{0, 1, 2, 3\}$ *of* $\mathfrak{h}_2(\mathbb{C})_u$*, with all three generators satisfying* $\eta L_{\mathrm{Mink}} + L_{\mathrm{Mink}}^T \eta = 0$. *This is the rotation subalgebra of the Lorentz group.*

(b) $\mathfrak{so}(6) \cong \mathfrak{su}(4)$ *acts on the internal W-directions. A rank obstruction prevents $\mathfrak{g}_{\mathrm{SM}}$ (rank 4) from embedding in $\mathfrak{su}(4)$ (rank 3); the reduction to the Standard Model gauge algebra requires the Todorov-Drenska intersection mechanism inside F_4 (see gap G8).*

(c) The cross-brackets $[\mathfrak{so}(3), \mathfrak{so}(6)] = 0$*: spacetime rotations and internal symmetries commute.*

(d) π_u is equivariant under all 18 generators (verified to precision 5×10^{-17}*).*

The compact Spin(9) cannot contain the non-compact boost generators of $SO(3,1)$. This is standard: gauge symmetries (compact) and spacetime symmetries (non-compact) belong to different sectors. Boosts arise from the spacetime metric $\det_2$ on $\mathfrak{h}_2(\mathbb{C})_u$ (which has $SL(2,\mathbb{C})$ isometry), not from the internal symmetry group. The rotation subalgebra $\mathfrak{so}(3)$ is the maximal compact subalgebra of $\mathfrak{so}(3,1)$.

4 The gravitational prepotential

The cubic determinant $\det(X)$ on $\mathfrak{h}_3(\mathbb{O})$ plays a dual role in the self-modeling framework: it appears as a factor in the experiential density ρ_J and as the gravitational prepotential determining all matter-gravity couplings. We prove that this double duty is forced by algebraic uniqueness.

4.1 The cubic norm

For $X \in \mathfrak{h}_3(\mathbb{O})$ as in Eq. (2), the cubic norm (determinant) is

$$\det(X) = \alpha\beta\gamma - \alpha|x_1|^2 - \beta|x_2|^2 - \gamma|x_3|^2 + 2\,\mathrm{Re}\big((x_1x_2)x_3\big). \tag{12}$$

This is the unique (up to scale) F_4-invariant cubic polynomial on $\mathfrak{h}_3(\mathbb{O})$.

Theorem 6 (Springer, 1962 [15])**.** $\dim \mathrm{Sym}^3(27^*)^{F_4} = 1$. *That is, the space of F_4-invariant cubic forms on $\mathfrak{h}_3(\mathbb{O})$ is one-dimensional.*

The F_4-invariance of $\det(X)$ is a classical consequence of Springer's theorem. As a consistency check, we have verified it computationally under three generating subgroups of F_4:

- S_3 (permutations of diagonal entries): 60 tests, maximum error 7.1×10^{-15}.
- G_2 (octonion automorphisms): 210 tests using 14 Schafer derivation generators [16], maximum error 2.0×10^{-13}.
- Spin(9) (Peirce representation): 360 tests using 36 grade-2 generators, maximum error 2.2×10^{-5} at $\epsilon = 10^{-6}$ (consistent with $O(\epsilon^2)$ truncation).

4.2 The d_{IJK} tensor

The cubic norm defines a totally symmetric tensor d_{IJK} in the Peirce basis via

$$\det(X) = d_{IJK}\, h^I h^J h^K, \tag{13}$$

where h^I are coordinates in the 27-dimensional Peirce basis ($I = 1$ for V_1; $I = 2, \ldots, 17$ for $V_{1/2}$; $I = 18, \ldots, 27$ for V_0). The index $I = 0$ is reserved for the graviphoton (from the gravity multiplet, not from the Jordan algebra); see Sec. 5.

The tensor d_{IJK} has 106 nonzero entries out of 3654 distinct triples (97% sparse), organized into exactly two nonzero Peirce blocks:

$$d_{IJK} \neq 0 \iff \begin{cases} (I, J, K) \in (V_1, V_0, V_0): & 10 \text{ entries,} \\ (I, J, K) \in (V_{1/2}, V_{1/2}, V_0): & 96 \text{ entries.} \end{cases} \tag{14}$$

All other Peirce block combinations are exactly zero (verified exhaustively, not sampled). The two nonzero blocks have clear physical interpretations:

- (V_1, V_0, V_0): the observer's coupling to the spacetime metric. The 10 entries reproduce the $\det_2$ bilinear form on V_0 exactly. This is the gravitational self-coupling.
- $(V_{1/2}, V_{1/2}, V_0)$: matter coupled to spacetime. The 96 entries split as 48 spacetime (coupling through the four $\mathfrak{h}_2(\mathbb{C})_u$ directions) plus 48 internal (coupling through the six W-directions). All 16 matter fields in $V_{1/2}$ couple to all 4 spacetime directions: the coupling is universal.

4.3 The double duty corollary

The cubic norm $\det(X)$ appears in two contexts within the self-modeling framework:

1. As a factor in the experiential density $\rho_J = \det(X) \cdot (\mathrm{Tr}(X^2) - \frac{1}{3})$, governing which states support self-modeling.
2. As the prepotential of the Gunaydin-Sierra-Townsend MESGT (Sec. 5), governing gravitational dynamics.

Corollary 7 (Double duty)**.** *Both roles of* $\det(X)$ *are forced by the same algebraic uniqueness constraint. Specifically:*

(a) The GST framework requires a cubic prepotential V *on* $\mathfrak{h}_3(\mathbb{O})$ *that is invariant under* $\mathrm{Aut}(\mathfrak{h}_3(\mathbb{O})) = F_4$*.*

(b) By Theorem 6, $\dim \mathrm{Sym}^3(27^*)^{F_4} = 1$*, so* $V = c \cdot \det(X)$ *for some constant c.*

(c) The experiential density ρ_J *also requires an* F_4*-invariant cubic factor, which is again* $\det(X)$*.*

Therefore both roles use $\det(X)$ *because it is the only cubic* F_4*-invariant available. The two roles enter with different physical interpretations (scalar function vs. prepotential) but share the same algebraic source. Neither is derived from the other.*

The proof is non-circular: the GST framework appears only as a *hypothesis* ("*if* a cubic prepotential is needed, it must be $\det(X)$"), not as a derivation step. The existence of the GST Lagrangian provides the hypothesis; the uniqueness of $\det(X)$ provides the conclusion.

4.4 Quantum numbers

The 16-dimensional $V_{1/2}$ subspace carries one generation of Standard Model fermions. The quantum number assignments (electric charge Q, hypercharge Y, isospin J_{3L}, J_{3R}, $B-L$) have been verified to match Paper 7 as a multiset, including color multiplicities (12 quarks + 4 leptons = 16 states). This confirms that the d_{IJK} coupling structure is compatible with the matter content derived independently from the Peirce decomposition and chirality chain.

5 Identification with exceptional supergravity

The algebraic data of $\mathfrak{h}_3(\mathbb{O})$ (the Peirce decomposition, the cubic norm $\det(X)$, and the $E_{6(-26)}$ structure group) uniquely match the exceptional entry in the Gunaydin-Sierra-Townsend (GST) classification of N=2 Maxwell-Einstein supergravity theories. This identification plays the same role as the Jordan-von Neumann-Wigner classification in Paper 5: the algebraic structure is proved, and a classification theorem identifies it with a known physical theory. The N=2 supersymmetric structure is *identified* via the GST bijection as the unique match to the algebraic data, not assumed as an input.

5.1 The GST classification

Gunaydin, Sierra, and Townsend [4, 5] (see also [17] for a modern review) proved that degree-3 Euclidean Jordan algebras with positive-definite trace form are in bijection with $N{=}2$ Maxwell-Einstein supergravity theories in five dimensions. Given such an algebra J with cubic norm $N(h)$, the 5-dimensional bosonic Lagrangian is uniquely determined:

$$e^{-1}\mathcal{L}_5 = -\frac{R}{2} - g_{ij}\,\partial_\mu\phi^i\,\partial^\mu\phi^j - \tfrac{1}{4}\,G_{IJ}\,F^I_{\mu\nu}\,F^{J\mu\nu} \\ - \tfrac{C_{IJK}}{24}\,\epsilon^{\mu\nu\rho\sigma\tau}\,F^I_{\mu\nu}\,F^J_{\rho\sigma}\,A^K_\tau\,, \tag{15}$$

where $C_{IJK} = \frac{1}{6}\,d_{IJK}$, the scalar metric g_{ij} and gauge kinetic matrix G_{IJ} are derived from the cubic norm N via very special real (VSR) geometry, and R is the Ricci scalar. The Einstein-Hilbert term $-R/2$ is part of the GST output: the bijection identifies the full bosonic Lagrangian, including gravity.

Proposition 8. *The algebra $\mathfrak{h}_3(\mathbb{O})$ satisfies all three GST hypotheses:*

(H1) ***Degree 3:*** *$\mathfrak{h}_3(\mathbb{O})$ has a cubic norm $\det(X)$ (Theorem 6).*

(H2) ***Formally real:*** *$\mathfrak{h}_3(\mathbb{O})$ is one of the five algebras in the Jordan-von Neumann-Wigner classification [9] that are formally real (equivalently: $X \circ X = 0$ implies $X = 0$).*

(H3) ***Positive-definite trace form:*** *The bilinear form $\mathrm{Tr}(X \circ X) = \alpha^2 + \beta^2 + \gamma^2 + 2(|x_1|^2 + |x_2|^2 + |x_3|^2) > 0$ for $X \neq 0$. On the 26-dimensional tangent space at the identity, all eigenvalues are positive (minimum eigenvalue $\frac{1}{4}$).*

By the GST bijection, $\mathfrak{h}_3(\mathbb{O})$ therefore uniquely determines a 5-dimensional $N{=}2$ MESGT: the exceptional magic supergravity. The 4-dimensional theory is obtained via the r-map (dimensional reduction on a circle) [18], giving a special Kähler scalar manifold

$$\mathcal{M} = \frac{E_{7(-25)}}{E_{6(-78)} \times \mathrm{U}(1)}, \quad \dim_{\mathbb{R}} = 54. \tag{16}$$

5.2 Field content

The 4-dimensional field content, organized by Peirce sector, is:

Peirce sector	I range	Vectors	Scalars
V_1 (observer)	1	1	2
$V_{1/2}$ (SM matter)	2–17	16	32
V_0 (spacetime + internal)	18–27	10	20
Graviphoton	0	1	0
Total		28	54

The 28 vectors include the graviphoton ($I = 0$) from the gravity multiplet and 27 matter vectors from the Peirce decomposition. The 54 real scalars parametrize the Kähler manifold $\mathcal{M}$ [19]. After dimensional reduction via the r-map, all 27 Jordan algebra directions contribute to these scalars, including the 6 internal W-directions from $\ker(\pi_u)$ and the 4 spacetime-identified directions of V_0. The W-directions are not lost in the reduction; they become internal scalar moduli of the 4-dimensional theory.

5.3 Coupling decomposition

The $C_{IJK} = \frac{1}{6}\, d_{IJK}$ tensor governs all matter-gravity couplings. Its 106 nonzero entries decompose into three physical channels:

1. **Gravitational self-coupling** (V_1, V_0, V_0): 10 entries, reproducing the $\det_2$ bilinear form on V_0. This couples the observer degree of freedom ($I = 1$) to the spacetime sector.

2. **Matter-spacetime coupling** ($V_{1/2}, V_{1/2}, V_0$ restricted to the 4 spacetime directions of $\mathfrak{h}_2(\mathbb{C})_u$): 48 entries. Per-index counts: indices 18 and 19 each couple to 16 matter fields; indices 20 and 27 each couple to 8.

3. **Matter-internal coupling** ($V_{1/2}, V_{1/2}, V_0$ restricted to the 6 internal W-directions): 48 entries, 8 per internal index.

The coupling is *universal*: every matter field in $V_{1/2}$ couples to every spacetime direction. This universality is a structural property of the d_{IJK} tensor, reflecting the algebraic coupling of all matter to the spacetime sector.

5.4 The 4-dimensional Lagrangian

The complete 4-dimensional bosonic Lagrangian, identified via the GST bijection and r-map, is [18, 20]:

$$e^{-1}\mathcal{L}_{\rm bos} = -\frac{R}{2} + g_{i\bar{j}}\,\partial_\mu z^i \partial^\mu \bar{z}^{\bar{j}} \\ + \operatorname{Im}(\mathcal{N}_{IJ})\,F^I_{\mu\nu}F^{J\mu\nu} + \operatorname{Re}(\mathcal{N}_{IJ})\,F^I_{\mu\nu}{*F}^{J\mu\nu}, \qquad (17)$$

where:

- $g_{i\bar{j}} = \partial_i\partial_{\bar{j}}K$ is the Kähler metric, with Kähler potential K derived from the prepotential $F(X) = d_{IJK}X^IX^JX^K/(6X^0)$, where $I, J, K = 1, \dots, 27$ run over the matter fields and X^0 is the projective coordinate (graviphoton section).
- $\mathcal{N}_{IJ}$ is the gauge kinetic matrix (period matrix), also derived from $F(X)$ via special Kähler geometry.
- R is the 4-dimensional Ricci scalar.

Remark 9 (What $\det(X)$ determines)*.* The prepotential $\det(X)$ determines the scalar manifold $\mathcal{M}$, the gauge kinetic matrix $\mathcal{N}_{IJ}$, and all matter-gravity couplings C_{IJK}. The Einstein-Hilbert term $-R/2$ is part of the identified MESGT Lagrangian: the GST bijection maps the algebraic data of $\mathfrak{h}_3(\mathbb{O})$ to the *complete* bosonic Lagrangian including the gravitational sector. The identification is with a theory that has gravity; gravity is not independently derived from $\det(X)$.

5.5 N=2 supersymmetry as a consequence

In the construction above, N=2 supersymmetry was not assumed. The bosonic Lagrangian is established in two logically independent steps: very special real (VSR) geometry gives the Lagrangian from the cubic prepotential without any supersymmetric input, and the GST classification then identifies the same Lagrangian as the bosonic sector of the exceptional N=2 MESGT.

Step 1: VSR geometry (no SUSY input). The chain from $\mathfrak{h}_3(\mathbb{O})$ to the bosonic Lagrangian is:

(i) $\mathfrak{h}_3(\mathbb{O})$ is a degree-3 Euclidean Jordan algebra (algebraic definition).

(ii) $F_4 = \mathrm{Aut}(\mathfrak{h}_3(\mathbb{O}))$ gives the **27** representation (representation theory).

(iii) $E_{6(-26)}$ structure group defines the scalar manifold (symmetric space theory).

(iv) The VSR metric g_{ij} is $E_{6(-26)}$-covariant (Schur's lemma).

(v) The cubic prepotential $V = C_{IJK}h^I h^J h^K$ is unique (Springer).

(vi) The gauge kinetic matrix G_{IJ} is uniquely fixed (VSR geometry [18]).

(vii) Gauge invariance determines the Chern-Simons coupling (gauge theory).

Steps (i)–(vii) produce the unique bosonic Lagrangian determined by the cubic prepotential $\det(X)$ within VSR geometry. No supersymmetry appears at any step. (VSR geometry was developed within the N=2 supergravity program [18]; the steps above use only the cubic-polynomial structure, not SUSY variations, but the framework's origin should be noted.)

Step 2: GST classification (N=2 identification). The GST classification [4, 5] is exhaustive within its category: every degree-3 formally real Jordan algebra with positive-definite trace form corresponds to a unique N=2 MESGT. The existence of this bijection was established by GST through explicit verification of N=2 supersymmetry closure. Since $\mathfrak{h}_3(\mathbb{O})$ satisfies all three GST hypotheses (Proposition 8), the GST bijection identifies the VSR Lagrangian from Step 1 as the bosonic sector of the exceptional N=2 MESGT.

The N=2 label appears only in Step 2, as the *output* of the classification, not as an input to any step that builds the Lagrangian. A circularity audit confirms that supersymmetry is absent from Steps (i)–(vii). To be precise: the N=2 label enters because the GST classification theorem classifies N=2 MESGTs; it is not derived from the self-modeling framework independently of this classification context. The VSR and GST routes give the same Lagrangian, confirming that the result is not an artifact of the N=2 framework.

5.6 Cosmological constant

The ungauged N=2 MESGT has an identically vanishing scalar potential $V(z, \bar{z}) = 0$ [20], giving $\Lambda = 0$ at the classical level. The observed cosmological constant $\Lambda > 0$ is not explained by this framework. Obtaining

$\Lambda \neq 0$ would require either gauging the theory (introducing a scalar potential) or invoking quantum corrections. This is an open limitation, shared with essentially all approaches to the cosmological constant problem.

6 Discussion

We have shown that the self-modeling basin $\mathfrak{h}_3(\mathbb{O})$, its Peirce decomposition, and its unique cubic invariant $\det(X)$ uniquely match the exceptional entry in the GST classification of $N=2$ Maxwell-Einstein supergravity theories. The complete bosonic Lagrangian, including the Einstein-Hilbert term $-R/2$, is identified with the algebraic data of $\mathfrak{h}_3(\mathbb{O})$ via this classification. This section provides an honest accounting of what is proved, what is identified, and what remains open.

6.1 What is proved

The following results are established by rigorous mathematical argument or verified computationally to machine precision:

1. $\mathfrak{h}_3(\mathbb{O})$ is the unique self-modeling basin (non-composability [3]).
2. The Peirce decomposition $27 = 1 \oplus 16 \oplus 10$ is a standard theorem of Jordan algebra theory.
3. The projection $\pi_u : \mathfrak{h}_2(\mathbb{O}) \to \mathfrak{h}_2(\mathbb{C})_u$ gives Minkowski signature $(+,-,-,-)$ from $\det_2$ (Proposition 1).
4. $\mathrm{KKT}(\mathfrak{h}_2(\mathbb{C})_u) = \mathfrak{so}(4,2)$ with 15 generators and Killing signature $(8,7)$ (Proposition 3).
5. $\det(X)$ is the unique F_4-invariant cubic form (Springer [15]).
6. d_{IJK} has exactly 106 nonzero entries in two Peirce blocks, with the coupling structure described in Sec. 4.
7. $\mathfrak{h}_3(\mathbb{O})$ satisfies all three GST hypotheses (Proposition 8).
8. Observer independence: all rank-1 idempotents give isomorphic Peirce decompositions (F_4 transitivity, Freudenthal [10]; Proposition 3).

6.2 What is identified

The following results depend on the GST classification theorem applied to the proved algebraic data:

9. The GST bijection identifies $\mathfrak{h}_3(\mathbb{O})$ with the exceptional 5-dimensional $N{=}2$ MESGT. The identification is uniquely determined by the three GST hypotheses.

10. The r-map (dimensional reduction on a circle) gives the 4-dimensional special Kähler theory with scalar manifold $E_{7(-25)}/(E_{6(-78)} \times \mathrm{U}(1))$.

11. The complete bosonic Lagrangian (17), including the Einstein-Hilbert term $-R/2$, is the output of the identification. All matter couplings are determined by $\det(X)$.

12. The $N{=}2$ structure appears as the output of the classification, not as an input (Sec. 5.5).

6.3 Weinberg consistency check

The identified theory is compatible with Weinberg's 1965 theorem [21]. Three of the four Weinberg hypotheses can be verified from the algebraic data of $\mathfrak{h}_3(\mathbb{O})$:

- **Lorentz invariance (W1):** The isometry group of $\det_2$ on $\mathfrak{h}_2(\mathbb{C})_u$ is $\mathrm{SL}(2,\mathbb{C}) \cong \mathrm{SO}(3,1)_0$, established in Sec. 3.

- **Masslessness (W3):** The ungauged MESGT has no Stückelberg mechanism and no Fierz-Pauli mass term.

- **Coupling universality:** All 16 matter fields in $V_{1/2}$ couple to all 4 spacetime directions via d_{IJK} (Sec. 4). This algebraic universality is a necessary condition for Weinberg's universal coupling (W4), but is not sufficient: full verification of W4 would require demonstrating coupling to the conserved stress-energy tensor $T_{\mu\nu}$ in the soft graviton limit.

The fourth hypothesis (spin-2, W2) is conditional on the manifold assumption (G0): perturbations of the metric on $\mathfrak{h}_2(\mathbb{C})_u \cong \mathbb{R}^{3,1}$ are symmetric rank-2 tensors. The Weinberg argument provides a consistency check on the identification, not an independent derivation of $-R/2$.

6.4 Gap inventory

We classify gaps by severity: *chain-critical* (would break the identification if fatal), *scope-limiting* (limit but do not break the chain), and *open* (no known mechanism within the current framework).

G0: Manifold assumption.
The entire identification requires that the algebraic data of $\mathfrak{h}_3(\mathbb{O})$ propagate on a smooth manifold as a dynamical field theory. This is the step from algebraic Minkowski space ($\mathfrak{h}_2(\mathbb{C})_u \cong \mathbb{R}^{3,1}$) to a spacetime manifold carrying fields. Without this assumption, the GST Lagrangian has no arena. This assumption is shared with every quantum gravity program and is stated explicitly in the introduction, but it is the single most important assumption in the paper.

Chain-critical gaps.

G1: V_0 = spacetime identification.
The algebraic $\mathfrak{h}_2(\mathbb{C})_u \cong \mathbb{R}^{3,1}$ has the correct signature, dimension, and conformal algebra. The identification of this algebraic space with physical spacetime is argued on the basis of matching structure, but is not proved from the self-modeling axioms alone. The entire gravitational interpretation depends on this identification. Additionally, the projection π_u is not a Jordan homomorphism (Proposition 1(d)), so the product structure of $\mathfrak{h}_2(\mathbb{O})$ is not preserved in the projected spacetime; this is part of the conceptual burden of the identification. While G0 is shared infrastructure with every quantum gravity program, G1 is specific to this framework and carries the greater conceptual burden: this is where the physical interpretation enters.

G2: 5d/4d dimensional mismatch.
The GST classification identifies a 5-dimensional MESGT from the full 27-dimensional algebra, while the Peirce-complement spacetime story extracts a 4-dimensional $\mathfrak{h}_2(\mathbb{C})_u$. The r-map connects these (dimensional reduction on a circle), but the coincidence that the algebraic spacetime dimension matches the dimensional reduction target is not explained by the framework.

Scope limitations.

G3: Compact $\mathfrak{so}(3)$ vs. non-compact $\mathfrak{so}(3,1)$.
The Spin(9) stabilizer contains only the compact rotation subalgebra

$\mathfrak{so}(3)$. The full Lorentz group $\mathrm{SL}(2,\mathbb{C})$ is recovered as the isometry group of $\det_2$ on $\mathfrak{h}_2(\mathbb{C})_u$ (a classical fact about 2×2 Hermitian matrices). Once the spacetime identification (G1) is granted, the Lorentz algebra follows from $\det_2$ as a standard classical fact; G3 is therefore subsumed by G1 and is not independently chain-critical.

G4: $\Lambda = 0$**.**
The ungauged MESGT gives $\Lambda = 0$ classically. The observed $\Lambda > 0$ is not explained.

G5: Bosonic sector only.
The Lagrangian (17) is the bosonic sector. The fermionic sector (gravitini, gaugini, hyperini) is predicted by the N=2 structure and determined by the same prepotential $\det(X)$, but has not been independently constructed from $\mathfrak{h}_3(\mathbb{O})$.

G6: Low-energy scope.
The identified Lagrangian is a classical supergravity theory. It does not constrain higher-derivative corrections or the UV completion.

G7: Minimal coupling.
The non-derivative stress-energy coupling $C_{i,j,a}$ is fully determined by d_{IJK}. Derivative terms follow from the standard minimal coupling prescription (unique for massless spin-2) but are not independently derived from $\mathfrak{h}_3(\mathbb{O})$.

Open problems.

G8: $\mathfrak{so}(6) \to \mathfrak{g}_{\mathrm{SM}}$**.**
The internal symmetry algebra $\mathfrak{so}(6) \cong \mathfrak{su}(4)$ (dimension 15, rank 3) does *not* contain $\mathfrak{g}_{\mathrm{SM}} = \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ (dimension 12, rank 4): a rank obstruction prevents the embedding. Todorov and Drenska [22] obtain G_{SM} as the intersection of Spin(9) with a second maximal-rank subgroup of F_4; embedding this mechanism within the self-modeling framework is an open problem.

G9: Three generations.
The Peirce decomposition gives one generation of SM fermions. The observed three generations are not explained. Boyle [23] obtains three generations from SO(8) triality acting on the complexified $\mathfrak{h}_3(\mathbb{O})$.

G10: Broader classification scope.

The GST classification operates within the framework of $N=2$ supergravity. The algebraic data of $\mathfrak{h}_3(\mathbb{O})$ uniquely match the exceptional entry in this classification. Whether a classification theorem exists for a broader class of theories that would also uniquely select this Lagrangian from the algebraic data alone, without reference to $N=2$ supersymmetry, is an open question. Very special real (VSR) geometry classifies the same cubic structures without mentioning supersymmetry; the VSR and GST classifications give the same Lagrangian, which provides evidence that the result is not an artifact of the $N=2$ framework.

6.5 Comparison with other approaches

Several groups have explored $\mathfrak{h}_3(\mathbb{O})$ as a foundation for particle physics. We compare the present work with three prominent approaches.

Farnsworth (2025) [8]. Farnsworth constructs a spectral triple over $\mathfrak{h}_3(\mathbb{O})$ within the Connes-Chamseddine framework, obtaining a $G_2 \times G_2$ gauge theory with charged scalar content. This provides a systematic noncommutative geometry framework for extracting gauge and scalar structure directly from $\mathfrak{h}_3(\mathbb{O})$. The present work addresses gravity, which Farnsworth does not. The two approaches are complementary.

Boyle (2020) [23]. Boyle observes that the complexified $\mathfrak{h}_3(\mathbb{O})$ encodes three generations of SM fermions via SO(8) triality. This is the only $\mathfrak{h}_3(\mathbb{O})$-based approach that addresses the generation problem. Gravity is not addressed.

Todorov-Drenska (2018–19) [22, 24]. Todorov and Drenska obtain the SM gauge group $G_{\text{SM}} = S(\text{U}(3) \times \text{U}(2))$ as the intersection of Spin(9) with a second maximal-rank subgroup of F_4. This group-theoretic result provides the mechanism needed to address gap G8, if it can be embedded in the self-modeling framework.

This work. The correspondence between $\mathfrak{h}_3(\mathbb{O})$ and the exceptional magic supergravity is a classical result of the GST program [4, 5]. The contribution of the present work is to reach this classical correspondence from the self-modeling framework: a companion paper [2] derives $\mathfrak{h}_3(\mathbb{O})$ from the self-modeling axiom (via C*-algebra structure and non-composability), whereas the other approaches take $\mathfrak{h}_3(\mathbb{O})$ as a given starting point. The Peirce decomposition then provides a spacetime interpretation via the C*-bottleneck, and the GST classification identifies the gravitational Lagrangian (not derived, but identified). To the author's knowledge, the present approach is the only $\mathfrak{h}_3(\mathbb{O})$-based framework that addresses gravity. It is possible that any of the other approaches could be extended to address gravity; this has not been done to date.

6.6 Relation to the lattice route

An earlier version of this paper (v1) attempted to derive Einstein's equations via a lattice Hamiltonian and the Jacobson entanglement equilibrium argument. That route was abandoned after failing peer review because the lattice is a modeling choice, not forced by the algebra. The present approach uses no lattice: spacetime is the Peirce complement, and the Lagrangian is identified via the GST classification and the cubic norm. The two routes share the starting point ($\mathfrak{h}_3(\mathbb{O})$ from self-modeling) but differ in every subsequent step.

6.7 Summary

From a single definition (faithful self-modeling) and the identification of $\mathfrak{h}_3(\mathbb{O})$ as the self-modeling basin, we identify the complete bosonic Lagrangian of the exceptional $N{=}2$ magic supergravity. The identification proceeds through the GST classification: $\mathfrak{h}_3(\mathbb{O})$ satisfies the three GST hypotheses, and the bijection uniquely determines the Lagrangian including the Einstein-Hilbert term $-R/2$.

The chain has honest gaps (Sec. 6.4). Two are chain-critical: the spacetime identification (G1) and the 5d/4d dimensional mismatch (G2). Two are open problems with no current mechanism: the gauge group reduction (G8) and three generations (G9). The framework does not explain the cosmological constant, the fermionic sector is predicted but not independently derived, and the identified Lagrangian is a classical theory. These limitations are stated because they are real.

What the framework *does* provide is a single algebraic starting point from which both quantum mechanics (via the JvNW classification) and the gravitational Lagrangian (via the GST classification) can be identified: the self-modeling definition forces the Jordan algebra $\mathfrak{h}_3(\mathbb{O})$, and $\mathfrak{h}_3(\mathbb{O})$ contains, within its cubic norm and Peirce decomposition, the algebraic data that the GST bijection maps to the complete bosonic gravitational Lagrangian coupled to Standard Model matter. The cubic form $\det(X)$ does double duty as the self-modeling density factor and the gravitational prepotential, forced by the Springer uniqueness theorem. This double duty is not a coincidence but an algebraic necessity: there is only one F_4-invariant cubic form on $\mathfrak{h}_3(\mathbb{O})$.

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