Sequential Products on Finite-Dimensional Euclidean Jordan Algebras: A Complete Classification

Bryan Ehrlich

Independent researcher, Trumbull, Connecticut 06611, USA

ehrlich.bryan@gmail.com

Abstract

A sequential product assigns to an ordered pair of effects (a, b) an effect $a \cdot b$, read "measure a, then b," subject to seven standard axioms, one of them norm-continuity in the first argument. We classify these products on every finite-dimensional Euclidean Jordan algebra. A product cannot couple distinct simple summands, so the answer is typewise. On real, quaternionic and exceptional summands of rank at least three, on real qubits, and on rank-one summands, the Lüders product $Q_{\sqrt{a}}\, b$ is forced. On each complex summand $M_n(\mathbb{C})^{\mathrm{sa}}$ with $n \geq 3$ the products are exactly Liu and Wu's twisted family $a^{1/2+it}\, b\, a^{1/2-it}$, one for each real t. On a rank-two spin factor $\mathbb{R} \oplus \mathbb{R}^d$ they correspond to the continuous odd fields that assign to each measurement axis an infinitesimal rotation fixing it; for the complex qubit this is a continuous function on $\mathbb{RP}^2$. The necessity proofs use only the axioms and classical Jordan structure. In rank at least three one argument, which does not use associativity of the coordinate algebra, shows that the residual freedom is generated by a central imaginary element of that algebra, which only $\mathbb{C}$ possesses. The surviving twist is the Lüders update followed by the flow that the measured effect generates through Alfsen and Shultz's dynamical correspondence. Hence a simple algebra of rank at least three carries a non-Lüders product exactly when it is the self-adjoint part of a C*-algebra. Every product is Lüders on sharp first effects, and the twist is detectable interferometrically with unsharp ones. The real and complex matrix cases are also proved in Lean 4.

1 Introduction

How far the operational axioms of Definition 2.1 determine a sequential product is a question going back to Gudder's open-problems list for sequential effect algebras [7]. Gudder and Latrémolière [9] proved that five conditions, one of which is a trace duality, characterize the Lüders product $a \bullet b = \sqrt{a}\, b\, \sqrt{a}$ (written with $\bullet$ throughout) on Hilbert-space effects. Without the trace duality the remaining conditions do not determine the product. Liu and Wu [14, Rem. 1] exhibited the one-parameter twisted family

$$a \circ_t b = a^{1/2+it}\, b\, a^{1/2-it}, \qquad t \in \mathbb{R}. \tag{1}$$

Every member satisfies all seven axioms and agrees with the Lüders product whenever the first argument is a projection (Shen–Wu [17, Thm. 3.1, Ex. 3.1]; Lemma 4.11 and corollary 6.2). Van de Wetering [19] showed that every finite-dimensional sequential-product space is order-isomorphic to a Euclidean Jordan algebra, also the setting of operational reconstructions of quantum theory [4]. What remained open is how much freedom the axioms leave on a given Euclidean Jordan algebra. Van de Wetering's uniqueness theorems [18] each add a selector to the axioms, and the constructions of Liu–Wu and Shen–Wu show that some freedom survives without one. This paper determines that freedom on every finite-dimensional Euclidean Jordan algebra.

Write $V_d = \mathbb{R} \oplus \mathbb{R}^d$ for the spin factor of vector dimension $d \geq 2$ (Section 2.2); $V_2 \cong M_2(\mathbb{R})^{\mathrm{sa}}$, $V_3 \cong M_2(\mathbb{C})^{\mathrm{sa}}$, $V_5 \cong M_2(\mathbb{H})^{\mathrm{sa}}$ and $V_9 \cong H_2(\mathbb{O})$. Let

$$\mathcal{A}_d = \big\{ A : S^{d-1} \to \mathfrak{so}(d) \text{ continuous} : A(v)v = 0,\ A(-v) = -A(v) \big\},$$

the continuous odd fields assigning to each unit vector v an infinitesimal rotation fixing v. For $A \in \mathcal{A}_d$, Definition 5.1 defines a product $\star_A$ on the effects of V_d; for an invertible effect with spectral data $a = \lambda_+ p_v + \lambda_- p_{-v}$ it is the Lüders update followed by the rotation $\exp(\log(\lambda_+/\lambda_-)\, A(v))$ of the hyperplane orthogonal to v.

Theorem 1 (Classification)**.** *Let $J = \bigoplus_{\alpha=1}^{m} J_\alpha$ be a finite-dimensional Euclidean Jordan algebra, decomposed into simple summands, and fix an isomorphism of each summand onto its model in the Jordan–von Neumann–Wigner list. Let $(\cdot)$ be any binary operation on the effects of J satisfying the seven axioms* (S1)–(S7) *of Definition 2.1. Then $a \cdot b = \sum_\alpha a_\alpha \cdot b_\alpha$, and on each summand the product is:*

- *the Lüders product $a \cdot b = Q_{\sqrt{a}}\, b$, if J_α is $\mathbb{R}$, $M_n(\mathbb{R})^{\mathrm{sa}}$ with $n \geq 2$, $M_n(\mathbb{H})^{\mathrm{sa}}$ with $n \geq 3$, or $H_3(\mathbb{O})$;*
- *$a \cdot b = a^{1/2+it_\alpha}\, b\, a^{1/2-it_\alpha}$ for a unique $t_\alpha \in \mathbb{R}$, if $J_\alpha = M_n(\mathbb{C})^{\mathrm{sa}}$ with $n \geq 3$ (with $0^{1/2\pm it} := 0$);*
- *$\star_{A_\alpha}$ for a unique $A_\alpha \in \mathcal{A}_d$, if $J_\alpha = V_d$ with $d \geq 3$.*

(The real qubit $V_2 \cong M_2(\mathbb{R})^{\mathrm{sa}}$ appears in the first line.) Conversely every independent choice of these parameters defines such a product, so that relative to the fixed identifications there is a bijection of sets

$$\mathcal{M}_{\mathrm{SP}}(J) \ \cong \prod_{\alpha\,:\,J_\alpha = M_n(\mathbb{C})^{\mathrm{sa}},\ n\geq 3} \mathbb{R} \ \times \prod_{\alpha\,:\,J_\alpha = V_d,\ d\geq 3} \mathcal{A}_d,$$

every other summand contributing a single point. Here $\mathcal{M}_{\mathrm{SP}}(J)$ denotes the set of (S1)–(S7) *products on the fixed (labelled) effect algebra of J (Remark 4.12). For the complex qubit, $\mathcal{A}_3 \cong C(\mathbb{RP}^2, \mathbb{R})$ (Corollary 5.3).*

The complex matrix algebras of rank at least three, the algebras of standard quantum mechanics, are the one case in which a single real parameter survives. We state the case of rank at least three separately.

Theorem 2 (Rank at least three)**.** *Let J be a simple Euclidean Jordan algebra of rank $n \geq 3$ and $(\cdot)$ a sequential product on its effects. If J is of real, quaternionic or exceptional type, $(\cdot)$ is the Lüders product. If $J = M_n(\mathbb{C})^{\mathrm{sa}}$, there is a unique $t \in \mathbb{R}$ with $a \cdot b = a^{1/2+it}\, b\, a^{1/2-it}$ for all effects a, b.*

The parameter t is stated for the fixed labelled algebra, on which it is unique; quotienting by all Jordan automorphisms identifies t with $-t$ through the transpose (Remark 4.12). Rank three is a sharp threshold in the complex series: on $M_2(\mathbb{C})^{\mathrm{sa}} = V_3$ the parameter becomes a continuous function of the measurement frame. In rank at least three, two frames that differ inside a rank-two block share a third atom, whose cross-coherence space compares their parameters (Lemma 4.7); a qubit frame has no third atom. In the spin factors of vector dimension $d \geq 4$, which include $M_2(\mathbb{H})^{\mathrm{sa}}$ and $H_2(\mathbb{O})$, the per-frame freedom is an element of $\mathfrak{so}(d-1)$, an infinitesimal rotation of the hyperplane orthogonal to the measurement axis; for $d = 3$ it is a single real number.

Dynamical reading. On $M_n(\mathbb{C})^{\mathrm{sa}}$, write $s(a)$ for the support projection of a and $U_a^t := (\mathbb{1} - s(a)) + \sum_{\lambda>0} \lambda^{it} p_\lambda$ for the unitary built from the *support* logarithm of a, so that $U_a^t = a^{it}$ when a is invertible. Then $a^{1/2+it} = \sqrt{a}\, U_a^t$ and

$$a \circ_t b = \sqrt{a}\,\big(U_a^t\, b\, (U_a^t)^*\big)\,\sqrt{a} = \mathrm{Ad}_{U_a^t}\big(a \bullet b\big), \tag{2}$$

with $\mathrm{Ad}_U(x) := UxU^*$. On effects (Heisenberg picture) the product is a Lüders update followed by conjugation by the outcome-conditioned unitary U_a^t. Dually, the unnormalized state operation is $\rho \mapsto (U_a^t)^*(\sqrt{a}\,\rho\,\sqrt{a})\,U_a^t$, of trace $\mathrm{tr}(a\rho)$. The unitary $a^{it} = e^{it \log a}$ is the time-t evolution generated by $\log a$. Alfsen and Shultz [2] axiomatized this dual role of observables, as generators of motion, under the name *dynamical correspondence*, and proved that a JB-algebra admits one if and only if it is the self-adjoint part of a C*-algebra. Combining their theorem with Theorem 2 gives the following corollary: in rank at least three the seven axioms determine the update rule up to this motion.

Theorem 3 (Dynamical form)**.** *Let J be a simple Euclidean Jordan algebra of rank at least three.*

(i) *J carries a sequential product other than the Lüders product if and only if J admits a dynamical correspondence, that is, if and only if J is the self-adjoint part of a C*-algebra.*

(ii) *On $J = M_n(\mathbb{C})^{\mathrm{sa}}$, let $\psi_h(x) = \frac{i}{2}(hx - xh)$ be the dynamical correspondence of the matrix product. The sequential products are exactly $a \cdot b = Q_{\sqrt{a}} \exp\big(2t\, \psi_{\log a}\big)\, b$ for invertible effects a, $t \in \mathbb{R}$, extended to singular a by continuity.*

The equivalence in (i) fails at rank two. The spin factors V_d with $d \geq 4$ admit no dynamical correspondence, yet they carry non-Lüders products (Remark 5.5); without frame overlap nothing forces the per-frame generators $A(v)$ to assemble into a global structure (Remark 6.1).

Every product is the Lüders product when its first argument is sharp (Corollary 6.2), so the projection postulate is forced and the freedom lies in unsharp measurements, where a single-qubit interferometric sequence detects it (Remark 6.3). If the product of a composite of complex systems restricts to the products of both factors, the factors share the composite's twist; a qubit that is a restricting factor of such a composite has a constant frame function (Remark 6.4).

Method. The necessity proofs use only the axioms, standard Jordan-algebra theory and elementary Lie theory; the constructive halves cite van de Wetering for the Lüders product and Shen and Wu for the twisted products. The main tool is van de Wetering's comparison map $\Theta_a = Q_{\sqrt{a}}^{-1} L_a$ [18, §5], with $L_a = a \cdot (\cdot)$; in finite dimension we derive the properties we need of it directly from the axioms. A short *frame calculus* (Lemma 3.5) shows that effects diagonal in a common system of orthogonal idempotents are compatible and multiply as Lüders. It gives L_a a positive inverse, so Θ_a is a unital order automorphism and hence a Jordan automorphism (Proposition 3.7). Restricted to the effects diagonal in one Jordan frame, $a \mapsto \Theta_a$ is a continuous homomorphism into the frame stabilizer, whose differential acts on each Peirce block V_{ij} as $(r_i - r_j)\, T_{ij}$ (Lemma 4.2). In coordinates, the derivation rule across three blocks forces T_{ij} to be multiplication by one central imaginary element of the coordinate algebra (Proposition 4.3). This disposes of the real, quaternionic and octonionic types at once and leaves one scalar per frame on the complex type, which frame overlap makes global. Rank two is treated separately by the cone geometry of the spin factors (Section 5).

Machine verification. Part of the classification is also proved in Lean 4. On the matrix carriers the formalization proves, from the seven axioms, that every product on $M_N(\mathbb{R})^{\mathrm{sa}}$, $N \geq 1$, is Lüders and that every product on $M_N(\mathbb{C})^{\mathrm{sa}}$, $N \geq 3$, is $\circ_t$ for a unique real t, both on all effects, singular ones included, and that each $\circ_t$ is a product. For the complex qubit it proves Corollary 5.3(ii): distinct continuous frame functions give products that differ on some pair of effects, and every product agrees on effects with the one of exactly one continuous function on $\mathbb{RP}^2$. The Lean proofs of these statements follow different arguments from the printed ones. The central-imaginary argument of Section 4 is also checked, for strictly positive first effects diagonal in the standard frames of $M_n(\mathbb{H})^{\mathrm{sa}}$ and $H_3(\mathbb{O})$, with the Archimedean property of the order as a hypothesis. The spin factors V_d with $d \geq 4$, Theorem 3, and the classification theorems over an abstract simple algebra are not formalized. Declaration names, commits and the axiom audit are in the Data Availability statement.

Organization. Section 2 fixes notation and the axioms. Section 3 derives the comparison automorphism from the axioms. Section 4 proves Theorem 2, Section 5 classifies the spin factors and assembles Theorem 1, and Section 6 proves Theorem 3 and the physical consequences. Section 7 compares with prior work.

2 Preliminaries

2.1 Euclidean Jordan algebras

A *Euclidean Jordan algebra* (EJA) is a finite-dimensional real commutative algebra $(J, \circ)$ with unit e satisfying the Jordan identity $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ and carrying an associative inner product $\langle x, y \rangle = \mathrm{tr}(x \circ y)$, where tr is the Jordan trace. We follow Faraut and Korányi [5]. The cone J^+ of squares is self-dual for $\langle \cdot, \cdot \rangle$, and for $x \geq 0$, $\mathrm{tr}\, x = 0$ only if $x = 0$. The *effects* are $[0, 1]_J = \{a : 0 \leq a \leq e\}$. The multiplication operator is $T_x(y) = x \circ y$ and the quadratic representation is $Q_x = 2T_x^2 - T_{x^2}$. For invertible $a \geq 0$, $Q_{\sqrt{a}}$ is a linear order

automorphism with inverse $Q_{a^{-1/2}}$ and $Q_{\sqrt{a}}e = a$ [5, Prop. III.2.2]. A Jordan automorphism g satisfies $gT_xg^{-1} = T_{gx}$ and hence $gQ_xg^{-1} = Q_{gx}$.

A *complete orthogonal system* (COS) is a finite family of nonzero idempotents $f_1, \dots, f_m$ with $f_k \circ f_l = 0$ for $k \neq l$ and $\sum_k f_k = e$. A *Jordan frame* is a COS of primitive idempotents; all Jordan frames of J have the same size, the rank of J. An element x is *diagonal in* a COS if $x = \sum_k \lambda_k f_k$. By the spectral theorem every element is diagonal in some Jordan frame. For an idempotent q the Peirce decomposition is $J = J_2(q) \oplus J_1(q) \oplus J_0(q)$ with $J_k(q) = \{x : q \circ x = \frac{k}{2}x\}$. Here $J_2(q)$ and $J_0(q)$ are EJAs with units q and $e - q$, and $J_2(q) \circ J_0(q) = 0$. For a Jordan frame $F = \{p_1, \dots, p_n\}$,

$$J = \bigoplus_i \mathbb{R}p_i \;\oplus\; \bigoplus_{i<j} V_{ij}, \qquad V_{ij} = \{x : p_i \circ x = \tfrac{1}{2}x = p_j \circ x\}. \tag{3}$$

If $x = \sum_k x_k p_k$ then Q_x acts by x_k^2 on $\mathbb{R}p_k$ and by $x_i x_j$ on V_{ij}; the same holds blockwise for a COS. When J is simple, $\dim V_{ij} = d$ is the Peirce multiplicity, all Jordan frames are conjugate under $\mathrm{Aut}(J)$ [5, Thm. IV.2.5], and $\mathrm{Aut}(J)$ is a compact Lie group with Lie algebra the derivations $\mathrm{Der}(J)$. The simple EJAs are $\mathbb{R}$; $M_n(\mathbb{K})^{\mathrm{sa}} = H_n(\mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $n \geq 3$; $H_3(\mathbb{O})$; and the spin factors V_d, $d \geq 2$, which are the simple EJAs of rank two [13, 5]. The rank-at-least-three types have Peirce multiplicity $1, 2, 4, 8$.

The standard (Lüders) product on J is $a \bullet b = Q_{\sqrt{a}}\, b$; on $M_n(\mathbb{C})^{\mathrm{sa}}$ it is $\sqrt{a}\, b\, \sqrt{a}$. Throughout, e, $\mathbb{1}$ and I denote the unit, and every functional-calculus expression takes the value 0 at eigenvalue 0, so $0^{1/2 \pm it} := 0$.

2.2 Spin factors

For $d \geq 2$ the spin factor is $V_d = \mathbb{R} \oplus \mathbb{R}^d$ with product $(\alpha, x) \circ (\beta, y) = (\alpha\beta + x \cdot y,\ \alpha y + \beta x)$ and unit $e = (1, 0)$. Its cone is $\{(\alpha, x) : \alpha \geq \|x\|\}$ and its order-unit norm is $\|(\alpha, x)\|_e = |\alpha| + \|x\|$, with $\|\cdot\|$ Euclidean. For a unit vector v put $p_v = \frac{1}{2}(1, v)$. The p_v are the primitive idempotents, the Jordan frames are the pairs $\{p_v, p_{-v}\}$, and the Peirce block of that frame is $\{0\} \oplus v^{\perp}$. Every nonscalar element is $\lambda_+ p_v + \lambda_- p_{-v}$ with $\lambda_+ > \lambda_-$ and a unique unit vector v, its *axis*. For $a = \lambda_+ p_v + \lambda_- p_{-v} \geq 0$ the map $Q_{\sqrt{a}}$ has multipliers λ_+, λ_- and $\sqrt{\lambda_+ \lambda_-}$ on $\mathbb{R}p_v$, $\mathbb{R}p_{-v}$ and $v^{\perp}$.

The Jordan automorphisms of V_d are exactly the maps $1 \oplus O$ with $O \in \mathrm{O}(d)$. Indeed an automorphism preserves the trace $\mathrm{tr}(\alpha, x) = 2\alpha$, hence maps the traceless part $\{0\} \oplus \mathbb{R}^d$ to itself. Since $(0, x) \circ (0, x) = (\|x\|^2, 0)$ it preserves the Euclidean norm there. Conversely each $1 \oplus O$ preserves the product.

The identification $(\alpha, x) \mapsto \alpha I + x \cdot \sigma$, with σ the Pauli vector, is a Jordan isomorphism $V_3 \cong M_2(\mathbb{C})^{\mathrm{sa}}$ carrying p_v to the projection $P_v = \frac{1}{2}(I + v \cdot \sigma)$ and the order-unit norm to the operator norm. Similarly $V_2 \cong M_2(\mathbb{R})^{\mathrm{sa}}$, $V_5 \cong M_2(\mathbb{H})^{\mathrm{sa}}$ and $V_9 \cong H_2(\mathbb{O})$.

2.3 Sequential products

Following Gudder–Greechie [8] and van de Wetering [19]:

Definition 2.1 (Sequential product space, [19, Def. 2])**.** Let $(V, \leq, \mathbb{1})$ be an order unit space with a binary operation $(\cdot) : [0,1]_V \times [0,1]_V \to [0,1]_V$. Write $a \mid b$ when $a \cdot b = b \cdot a$. Then $(V, \cdot)$ is a *sequential product space* if:

(S1) **Additivity:** $a \cdot (b + c) = a \cdot b + a \cdot c$ whenever $b + c \leq \mathbb{1}$.

(S2) **Continuity:** $a \mapsto a \cdot b$ is continuous in the order unit norm.

(S3) **Unitality:** $\mathbb{1} \cdot a = a$ for all a.

(S4) **Symmetry of orthogonality:** $a \cdot b = 0 \Rightarrow b \cdot a = 0$.

(S5) **Associativity of compatible effects:** $a \mid b \Rightarrow a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

(S6) **Additivity of compatibility:** $a \mid b \Rightarrow a \mid (\mathbb{1} - b)$; if also $a \mid c$ then $a \mid (b + c)$ when $b + c \leq \mathbb{1}$.

(S7) **Multiplicativity of compatibility:** $a \mid b$ and $a \mid c \Rightarrow a \mid b \cdot c$.

These are van de Wetering's Definition 2 [19] with the domain riders made explicit; (S2) is his norm-continuity in the first argument, which he adds to the sequential-effect-algebra list. His structure theorem places the problem on Euclidean Jordan algebras.

Theorem 2.2 ([19, Thm. 1])**.** *A finite-dimensional sequential product space is order-isomorphic to a Euclidean Jordan algebra.*

The identification is unital, since van de Wetering's Jordan product takes the order unit as its unit, so the axioms and the effect interval transport to J. We therefore fix a finite-dimensional EJA J and study the operations on $[0, 1]_J$ satisfying (S1)–(S7); we call them *sequential products on* J and write $L_a(b) = a \cdot b$.

Lemma 2.3 (Effects span)**.** *In an order unit space, $[0, 1]_V$ contains the closed ball of radius $\frac{1}{2}$ about $\frac{1}{2}\mathbb{1}$, hence spans V and has nonempty interior; two linear maps agreeing on $[0, 1]_V$ are equal.*

Proof. If $\|v\| \leq \frac{1}{2}$ then $0 \leq \frac{1}{2}\mathbb{1} + v \leq \mathbb{1}$. □

3 From the Axioms to a Jordan Automorphism

Fix a finite-dimensional EJA J and a sequential product $(\cdot)$ on J. This section derives, from (S1)–(S7) alone, that for every invertible effect a the map $\Theta_a = Q_{a^{-1/2}} L_a$ is a Jordan automorphism with the properties the classification needs. The map is van de Wetering's comparison map [18, §5]. His construction works in a general sequential effect algebra and reaches it through normality and pseudo-inverse hypotheses; in a finite-dimensional EJA the spectral theorem makes a direct route available.

3.1 Linear extension and scalars

Lemma 3.1 (Positive linear extension)**.** *For each effect a:* (i) *L_a extends uniquely to a positive linear map $J \to J$;* (ii) *$a \cdot 0 = 0 \cdot a = 0$ and $a \cdot \mathbb{1} = \mathbb{1} \cdot a = a$, so $L_a(\mathbb{1}) = a$ and $L_a([0, 1]_J) \subseteq [0, a]$;* (iii) *$\|L_a\|_{e \to e} = \|a\|_e$.*

Proof. (i) L_a is additive by (S1) and maps $[0,\mathbb{1}]$ into $[0,\mathbb{1}]$. For $x \geq 0$ with $x/n \leq \mathbb{1}$ set $\widetilde{L}_a(x) := n\, L_a(x/n)$; additivity makes this independent of n and additive on J^+, order boundedness gives real homogeneity, and extension by differences yields a unique positive linear map on $J = \mathrm{span}[0,\mathbb{1}]_J$ (Lemma 2.3). (ii) (S1) with $b = c = 0$ gives $a \cdot 0 = 0$, and (S4) gives $0 \cdot a = 0$, so $a \mid 0$. (S6) gives $a \mid \mathbb{1}$, and (S3) gives $a \cdot \mathbb{1} = \mathbb{1} \cdot a = a$. Monotonicity of L_a gives $L_a([0,\mathbb{1}]_J) \subseteq [0,a]$. (iii) $-\mathbb{1} \leq x \leq \mathbb{1}$ implies $-a \leq L_a x \leq a$, and $L_a\mathbb{1} = a$. □

Lemma 3.2 (Scalars)**.** *For every effect a and every $q \in [0,1]$: $a \mid q\mathbb{1}$, $L_{q\mathbb{1}} = qI$, and $L_{qa} = q\, L_a$.*

Proof. Fix $n \geq 1$. Trivially $a/n \mid a/n$, and repeated (S6) addition gives $a/n \mid ka/n$ for $k \leq n$, every partial sum being at most $a \leq \mathbb{1}$; so $a/n \mid a$. Likewise $(\mathbb{1}-a)/n \mid (\mathbb{1}-a)$, and (S6) complementation gives $(\mathbb{1}-a)/n \mid a$. By symmetry of $\mid$ and (S6) addition, $a \mid a/n + (\mathbb{1}-a)/n = \mathbb{1}/n$. Hence $(\mathbb{1}/n) \cdot a = a \cdot (\mathbb{1}/n) = a/n$ by linearity of L_a, and repeating the addition, $a \mid k\mathbb{1}/n$ with $(k\mathbb{1}/n) \cdot a = ka/n$. So $L_{q\mathbb{1}} = qI$ on $[0,\mathbb{1}]_J$ for rational $q \in [0,1]$, and (S2) extends this to real q. Then $a \cdot (q\mathbb{1}) = qa = (q\mathbb{1}) \cdot a$, so $a \mid q\mathbb{1}$. Finally (S5) applied to $a \mid q\mathbb{1}$ gives, for every effect c, $a \cdot (qc) = a \cdot ((q\mathbb{1}) \cdot c) = (a \cdot q\mathbb{1}) \cdot c = (qa) \cdot c$, that is, $L_{qa} = qL_a$. □

Lemma 3.3 (Continuity)**.** *$a \mapsto L_a$ is continuous from $[0,\mathbb{1}]_J$ to the operators on J.*

Proof. By Lemma 2.3, $[0,\mathbb{1}]_J$ contains a basis $b_1, \ldots, b_N$ of J. (S2) makes each $a \mapsto L_a b_k$ continuous, and J is finite-dimensional. □

3.2 Frame calculus

Lemma 3.4. *If p, q are orthogonal idempotents, then $[0,p] \cap [0,q] = \{0\}$.*

Proof. Let $0 \leq x \leq p$ and $0 \leq x \leq q$. By self-duality, $0 \leq \langle x, e-p\rangle \leq \langle p, e-p\rangle = \mathrm{tr}(p-p^2) = 0$, so $\langle x, e-p\rangle = 0$; likewise $\langle x, e-q\rangle = 0$. As $p+q \leq e$, also $\langle x, e-p-q\rangle \geq 0$. Hence $0 = \langle x, (e-p)+(e-q)\rangle = \mathrm{tr}\, x + \langle x, e-p-q\rangle \geq \mathrm{tr}\, x \geq 0$, so $\mathrm{tr}\, x = 0$ and $x = 0$. □

Lemma 3.5 (Frame calculus)**.** *Let $\{f_1, \ldots, f_m\}$ be a complete orthogonal system and let $a = \sum_k \lambda_k f_k$ and $b = \sum_k \mu_k f_k$ be effects, so that all $\lambda_k, \mu_k \in [0,1]$ (apply the positive map Q_{f_k} to $0 \leq a \leq e$). Then $a \mid b$ and*

$$a \cdot b = \sum_k \lambda_k \mu_k f_k = Q_{\sqrt{a}}\, b.$$

Proof. Step 1: $L_{f_i} f_j = \delta_{ij} f_i$. (S6) applied to $f_i \mid f_i$ gives $f_i \mid (e - f_i)$. The common value $f_i \cdot (e-f_i) = (e-f_i) \cdot f_i$ lies in $[0,f_i] \cap [0, e-f_i]$ by Lemma 3.1(ii), hence vanishes by Lemma 3.4. For $j \neq i$, $0 \leq f_j \leq e - f_i$, so $0 \leq L_{f_i} f_j \leq L_{f_i}(e-f_i) = 0$. Then $L_{f_i} f_i = L_{f_i} e - \sum_{j \neq i} L_{f_i} f_j = f_i$.

Step 2: $f_i \mid a$. For $j \neq i$, $f_i \cdot f_j = 0$, so $f_j \cdot f_i = 0$ by (S4). For $\lambda \in [0,1]$ and every j, $f_i \cdot (\lambda f_j) = \lambda\, f_i \cdot f_j$ by linearity and $(\lambda f_j) \cdot f_i = \lambda\, f_j \cdot f_i$ by Lemma 3.2; so $f_i \mid \lambda f_j$. Every partial sum of $\sum_k \lambda_k f_k$ is at most $a \leq e$, the omitted terms being positive, so (S6) addition gives $f_i \mid a$.

Step 3: $a \mid b$. By symmetry $a \mid f_k$ for each k; as in Step 2, $a \mid \mu_k f_k$, and (S6) addition gives $a \mid b$.

Step 4: the value. By Step 1, $a \cdot f_k = f_k \cdot a = \sum_l \lambda_l L_{f_k} f_l = \lambda_k f_k$, so $a \cdot b = \sum_k \mu_k\, a \cdot f_k = \sum_k \lambda_k \mu_k f_k$, which is $Q_{\sqrt{a}}\, b$. □

The reduction to a simple algebra also follows from the axioms.

Proposition 3.6 (Central decomposition)**.** *Let $J = \bigoplus_{\alpha=1}^{m} J_\alpha$ be any finite-dimensional EJA with central summand units e_α, and let $(\cdot)$ be a sequential product on J. For an effect a write $a_\alpha := e_\alpha \circ a \in [0,1]_{J_\alpha}$. Then*

$$a \cdot b = \sum_{\alpha=1}^{m} a_\alpha \cdot b_\alpha, \qquad a_\alpha \cdot b_\alpha \in J_\alpha,$$

and each restriction is a sequential product on the order-unit space $(J_\alpha, \leq, e_\alpha)$. Conversely, given sequential products $(\cdot)_\alpha$ on the summands, $a \cdot b := \sum_\alpha (a_\alpha \cdot_\alpha b_\alpha)$ is a sequential product on J restricting to them.

Proof. Every primitive idempotent of J lies in one summand, so a Jordan frame of J in which a is diagonal refines the COS $\{e_\alpha\}$. By Lemma 3.5, $a \mid e_\alpha$ and $a \cdot e_\alpha = Q_{\sqrt{a}} e_\alpha = a_\alpha$. Likewise, for $b_\alpha \in [0, e_\alpha]$, a frame of J_α diagonalizing b_α together with frames of the other summands is a Jordan frame of J, so $e_\alpha \cdot b_\alpha = b_\alpha$. By (S5),

$$a \cdot b_\alpha = a \cdot (e_\alpha \cdot b_\alpha) = (a \cdot e_\alpha) \cdot b_\alpha = a_\alpha \cdot b_\alpha,$$

and $0 \leq a \cdot b_\alpha \leq a \cdot e_\alpha = a_\alpha \leq e_\alpha$ puts it in J_α, because the order of $J = \bigoplus_\beta J_\beta$ is componentwise, so $0 \leq x \leq e_\alpha$ forces every component of x outside J_α to vanish. Summing over α by (S1) gives the formula. Block-diagonality makes compatibility componentwise: for $x, y \in [0,1]_{J_\alpha}$, $\mathbb{1} - y = (e_\alpha - y) + \sum_{\beta \neq \alpha} e_\beta$, and the component formula gives $x \cdot (\mathbb{1} - y) = x \cdot (e_\alpha - y)$ and $(\mathbb{1} - y) \cdot x = (e_\alpha - y) \cdot x$. So the complement clause of (S6) restricts to each summand; the other clauses restrict directly, and the order-unit norm of J_α is the restriction of that of J. For the converse, effects, order and complements are componentwise, giving (S1), (S3), (S4). Compatibility holds in J exactly when it holds in every coordinate, so (S5)–(S7) reduce to the summands, and (S2) holds because the direct-sum order-unit norm is $\max_\alpha \| \cdot \|_\alpha$. □

3.3 The comparison automorphism

Proposition 3.7 (Comparison automorphism)**.** *For every invertible effect a, the map $\Theta_a := Q_{a^{-1/2}} L_a$ is a Jordan automorphism of J, so that*

$$a \cdot b = Q_{\sqrt{a}}\, \Theta_a(b) \qquad \text{for all effects } b.$$

Moreover:

(i) (Fixing) $\Theta_a y = y$ *for every y diagonal in a COS in which a is diagonal;*

(ii) (Continuity) $a \mapsto \Theta_a$ *is continuous on the invertible effects;*

(iii) (Cocycle) *if a, a' are invertible effects diagonal in a common COS, then* $\Theta_{a\circ a'} = \Theta_a\Theta_{a'} = \Theta_{a'}\Theta_a$.

Proof. Let $a = \sum_k \lambda_k p_k$ in a Jordan frame, all $\lambda_k > 0$. Put $\mu := \max_k \lambda_k^{-1} \geq 1$ and $b := \mu^{-1}a^{-1} = \sum_k (\mu\lambda_k)^{-1} p_k$, an effect. By Lemma 3.5, $a \mid b$ and $a \cdot b = b \cdot a = \mathbb{1}/\mu$. By (S5) and Lemma 3.2, for every effect c, $a \cdot (b \cdot c) = (\mathbb{1}/\mu) \cdot c = c/\mu$, and symmetrically. So $L_a L_b = L_b L_a = \mu^{-1} I$ on $[0,1]_J$, hence on J, and L_a is a linear bijection with positive inverse μL_b. Together with the corresponding facts for $Q_{\sqrt{a}}$, this makes Θ_a a linear bijection with $\Theta_a(J^+) = J^+$, and $\Theta_a \mathbb{1} = Q_{a^{-1/2}} a = \mathbb{1}$. A unital linear order automorphism of a Euclidean Jordan algebra is a Jordan automorphism (van Imhoff–Roelands [21, Cor. 2.5]; classically Alfsen–Shultz [3]). So $\Theta_a \in \mathrm{Aut}(J)$.

(i) If a and y are diagonal in a COS $\{f_k\}$, each f_k is an effect diagonal in that COS, so Lemma 3.5 gives $L_a f_k = Q_{\sqrt{a}} f_k$, that is $\Theta_a f_k = f_k$; and y is a linear combination of the f_k. (ii) Lemma 3.3 and continuity of $a \mapsto Q_{a^{-1/2}}$ on invertible elements. (iii) Lemma 3.5 gives $a \mid a'$ and $a \cdot a' = Q_{\sqrt{a}} a' = a \circ a'$, so (S5) gives $L_a L_{a'} = L_{a\circ a'}$ on $[0,1]_J$, hence on J. By (i), Θ_a fixes $\sqrt{a'}$, so $\Theta_a Q_{\sqrt{a'}} = Q_{\Theta_a \sqrt{a'}} \Theta_a = Q_{\sqrt{a'}} \Theta_a$. Hence

$$\Theta_{a\circ a'} = Q_{(a\circ a')^{-1/2}}\, Q_{\sqrt{a}}\, \Theta_a\, Q_{\sqrt{a'}}\, \Theta_{a'} = Q_{(a\circ a')^{-1/2}}\, Q_{\sqrt{a}}\, Q_{\sqrt{a'}}\, \Theta_a\, \Theta_{a'} = \Theta_a \Theta_{a'},$$

the three quadratic maps being diagonal on the common Peirce decomposition with multipliers that multiply. Exchanging the roles of a and a' gives $\Theta_{a'}\Theta_a$. □

Lemma 3.8 (Coalescence)**.** *Let q be an idempotent and $a = \lambda q + a_0$ an invertible effect with $\lambda > 0$ and $a_0 \in J_0(q)$. Then Θ_a is the identity on $J_2(q)$.*

Proof. Let $y \in J_2(q)$. By the spectral theorem in $J_2(q)$ and in $J_0(q)$, $y = \sum_\alpha \mu_\alpha e_\alpha$ and $a_0 = \sum_\beta \nu_\beta g_\beta$ with $\{e_\alpha\}$ a COS of $J_2(q)$ summing to q and $\{g_\beta\}$ a COS of $J_0(q)$ summing to $e - q$. As $J_2(q) \circ J_0(q) = 0$, $\{e_\alpha\} \cup \{g_\beta\}$ is a COS of J in which both y and $a = \sum_\alpha \lambda e_\alpha + \sum_\beta \nu_\beta g_\beta$ are diagonal. Apply Proposition 3.7(i). □

4 Rank at Least Three

4.1 Frame reduction

Fix a simple EJA J of rank $n \geq 3$, a Jordan frame $F = \{p_1, \ldots, p_n\}$, and for $r \in (-\infty, 0]^n$ the invertible effect $a(r) = \sum_k e^{r_k} p_k$. Write $\Theta_r := \Theta_{a(r)}$, and let $\mathrm{Stab}(F) = \{g \in \mathrm{Aut}(J) : gp_i = p_i \text{ for all } i\}$, the pointwise frame stabilizer, a closed subgroup of $\mathrm{Aut}(J)$.

Lemma 4.1 (Frame fixing)**.** *Each Θ_r lies in the identity component $\mathrm{Stab}(F)^\circ$ and preserves every Peirce block V_{ij}. In particular $L_{a(r)} = Q_{\sqrt{a(r)}} \Theta_r$ is block-diagonal for* (3)*.*

Proof. Each p_i is diagonal in F together with $a(r)$, so $\Theta_r p_i = p_i$ by Proposition 3.7(i). A Jordan automorphism fixing p_i and p_j preserves their joint $\frac{1}{2}$-eigenspace V_{ij}. By Proposition 3.7(ii) the path $s \mapsto \Theta_{sr}$, $s \in [0,1]$, joins $\Theta_0 = \mathrm{id}$ to Θ_r inside $\mathrm{Stab}(F)$. □

Lemma 4.2 (Diagonal homomorphism and hyperplane factorization)**.** *The map $r \mapsto \Theta_r$ extends to a continuous homomorphism $\chi : (\mathbb{R}^n, +) \to \mathrm{Stab}(F)^\circ$. Its differential $r \mapsto D_r := d\chi(r)$ is real-linear, each D_r is a derivation of J with $D_r p_k = 0$ and $D_r V_{ij} \subseteq V_{ij}$, and there are linear maps $T_{ij} : V_{ij} \to V_{ij}$ with*

$$D_r\big|_{V_{ij}} = (r_i - r_j)\, T_{ij} \qquad \text{for all } r \in \mathbb{R}^n. \tag{4}$$

Proof. For $r, r' \le 0$, $a(r) \circ a(r') = a(r+r')$, so $\Theta_{r+r'} = \Theta_r \Theta_{r'} = \Theta_{r'} \Theta_r$ by Proposition 3.7(iii); the image is abelian. Extend to $\mathbb{R}^n = S - S$, $S = (-\infty, 0]^n$, by $\chi(s - s') := \Theta_s \Theta_{s'}^{-1}$. This is well defined: if $s - s' = u - u'$ then $s + u' = u + s'$, so $\Theta_s \Theta_{u'} = \Theta_u \Theta_{s'}$ and, the image being abelian, $\Theta_s \Theta_{s'}^{-1} = \Theta_u \Theta_{u'}^{-1}$. It is continuous through the continuous decomposition $x = s(x) - s'(x)$ with $s_i(x) = \min(x_i, 0)$ and $s'_i(x) = \min(-x_i, 0)$. A continuous homomorphism between finite-dimensional Lie groups is smooth [11, Chs. 2–3], so $d\chi$ is real-linear, and a continuous homomorphism out of $\mathbb{R}^n$ is determined by its differential. Each D_r lies in the Lie algebra of $\mathrm{Stab}(F)$, which consists of derivations annihilating every p_k; such a derivation preserves each V_{ij}, since $p_i \circ x = \frac{1}{2} x$ implies $p_i \circ D_r x = \frac{1}{2} D_r x$. For $h \in S$ with $h_i = h_j$, Lemma 3.8 applied with $q = p_i + p_j$ gives $\chi(sh)|_{V_{ij}} = \mathrm{id}$ for all $s \ge 0$; differentiating at $s = 0$ gives $D_h|_{V_{ij}} = 0$. Differences of such h span the hyperplane $\{r_i = r_j\}$: any v with $v_i = v_j$ is $v = h - h'$ with $h = v - c(1, \dots, 1)$ and $h' = -c(1, \dots, 1)$ for $c \ge \max(0, \max_k v_k)$. So the linear map $r \mapsto D_r|_{V_{ij}}$ vanishes on that hyperplane and is a multiple of $r_i - r_j$. □

4.2 The central-imaginary lemma

By the Jordan–von Neumann–Wigner classification we may take $J = H_n(\mathbb{K})$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, or $J = H_3(\mathbb{O})$, and F the diagonal matrix units $p_k = E_{kk}$; a Jordan isomorphism transports sequential products to sequential products, so there is no loss of generality. For $i \ne j$ and $x \in \mathbb{K}$ put

$$x[ij] := x E_{ij} + \bar{x} E_{ji} \in V_{ij},$$

so that $x \mapsto x[ij]$ is a real-linear bijection $\mathbb{K} \to V_{ij}$ and $x[ji] = \bar{x}[ij]$. Two products are needed. For distinct i, j, k and $x, y, \gamma \in \mathbb{K}$,

$$x[ij] \circ y[jk] = \tfrac{1}{2}\,(xy)[ik], \qquad \gamma[ij] \circ 1[ij] = \mathrm{Re}(\gamma)\,(p_i + p_j). \tag{5}$$

For the first, $x[ij]\, y[jk] = xy\, E_{ik}$ and $y[jk]\, x[ij] = \bar{y}\bar{x}\, E_{ki} = \overline{xy}\, E_{ki}$, all other products of matrix units vanishing because i, j, k are distinct. For the second, $\gamma[ij]\, 1[ij] = \gamma E_{ii} + \bar{\gamma} E_{jj}$ and $1[ij]\, \gamma[ij] = \bar{\gamma} E_{ii} + \gamma E_{jj}$. Each entry involved is a single product of two elements of $\mathbb{K}$, so (5) holds for the octonions as well. In these coordinates (4) reads

$$D_r\big(x[ij]\big) = (r_i - r_j)\, t_{ij}(x)[ij] \qquad \text{for all } r \in \mathbb{R}^n,\ x \in \mathbb{K}, \tag{6}$$

for real-linear maps $t_{ij} : \mathbb{K} \to \mathbb{K}$.

Proposition 4.3 (Central-imaginary generator)**.** *There is a single $\gamma \in \mathbb{K}$ with $\gamma x = x\gamma$ for all $x \in \mathbb{K}$ and $\mathrm{Re}\,\gamma = 0$, such that $t_{ij}(x) = \gamma x$ for every ordered pair $i \ne j$.*

Proof. Fix distinct i, j, k. Apply the derivation D_r to the first identity of (5), using (6):

$$(r_i - r_k)\, t_{ik}(xy) = (r_i - r_j)\, t_{ij}(x)\, y + (r_j - r_k)\, x\, t_{jk}(y) \qquad \text{for all } r,\ x,\ y.$$

At $r = \epsilon_i$, the i-th standard basis vector, this reads $t_{ik}(xy) = t_{ij}(x)\, y$; at $r = \epsilon_k$ it reads $t_{ik}(xy) = x\, t_{jk}(y)$. Setting $y = 1$ in the first and $x = 1$ in the second gives $t_{ik} = t_{ij}$ and $t_{ik} = t_{jk}$. Call the common map t and put $\gamma := t(1)$. Setting $x = 1$ in the first gives $t(y) = \gamma y$, and setting $y = 1$ in the second gives $t(x) = x\gamma$. So γ is central and $t_{ij} = t_{jk} = t_{ik} = \gamma\,\cdot$.

The triple identity gives $t_{ij} = t_{ik}$ and $t_{ij} = t_{jk}$ whenever i, j, k are distinct. For fixed i and distinct $j, j' \neq i$, the triple (i, j, j') gives $t_{ij} = t_{ij'}$, so $t_{ij} =: s_i$ depends only on i. For $i \neq j$ choose a third index k; then $t_{ij} = t_{jk}$ reads $s_i = s_j$. So all t_{ij} coincide.

For the real part, take $x = 1[ij]$ and $r = \epsilon_i$, so that $D_r x = \gamma[ij]$. By (5), $x \circ x = p_i + p_j$, which D_r annihilates. The derivation rule gives $0 = D_r(x \circ x) = 2\, D_r x \circ x = 2\,\mathrm{Re}(\gamma)(p_i + p_j)$. □

The argument does not use associativity of $\mathbb{K}$ or the structure of $\mathrm{Stab}(F)$, so it applies to $H_3(\mathbb{O})$. The imaginary central elements are 0 for $\mathbb{R}$, $\mathbb{H}$ and $\mathbb{O}$ and $i\mathbb{R}$ for $\mathbb{C}$. (In $\mathbb{H}$ and $\mathbb{O}$ the centre is $\mathbb{R}$. For $\mathbb{O}$ directly: if γ is imaginary and central, then for imaginary x the relation $\gamma x + x\gamma = -2\langle \gamma, x\rangle$ gives $\gamma x = -\langle \gamma, x\rangle$, and a unit imaginary x orthogonal to γ gives $\|\gamma\| = \|\gamma x\| = 0$.)

Proposition 4.4 (One generator per frame)**.** *Let $n \geq 3$ and let F be any Jordan frame of the simple algebra J of rank n.*

(i) *If J is of real, quaternionic or exceptional type, $\Theta_a = \mathrm{id}$ for every invertible effect a diagonal in F.*

(ii) *If $J = M_n(\mathbb{C})^{\mathrm{sa}}$, there is a unique $t_F \in \mathbb{R}$ with $\Theta_a = \mathrm{Ad}_{a^{it_F}}$, that is $a \cdot b = a^{1/2+it_F}\, b\, a^{1/2-it_F}$, for every invertible effect a diagonal in F and every effect b.*

Proof. Let $g \in \mathrm{Aut}(J)$ carry the standard frame F_0 to F as a set, and in case (ii) take $g = \mathrm{Ad}_U$ with U unitary, which is possible because orthonormal bases are unitarily conjugate. The operation $a \cdot' b := g^{-1}(ga \cdot gb)$ is again a sequential product, since g is a unital order automorphism preserving the order-unit norm, and its comparison maps are $\Theta'_a = g^{-1}\Theta_{ga} g$. So it suffices to treat $F = F_0$, where every invertible effect diagonal in F_0 is some $a(r)$. In case (i), Proposition 4.3 gives $\gamma = 0$, so D_r vanishes on every V_{ij}; it also annihilates every p_k, so $D_r = 0$ by (3), hence $\Theta_r = \chi(r) = \mathrm{id}$. In case (ii) it gives $\gamma = it_F$ with t_F real, so D_r multiplies the coordinate of V_{ij} by $it_F(r_i - r_j)$ and $\Theta_r = \exp D_r$ multiplies it by $e^{it_F(r_i - r_j)}$. That is $\mathrm{Ad}_{a(r)^{it_F}}$, because $a(r)^{it_F} = \mathrm{diag}(e^{it_F r_k})$ fixes the diagonal and conjugates the (i, j) entry by $e^{it_F r_i}(\cdot)e^{-it_F r_j}$. Transporting back by $g = \mathrm{Ad}_U$ gives $\Theta_{ga} = \mathrm{Ad}_U\, \mathrm{Ad}_{a^{it_F}}\, \mathrm{Ad}_U^{-1} = \mathrm{Ad}_{(ga)^{it_F}}$. The parameter is unique because $\mathrm{Ad}_{a^{it}}$ determines t as a ranges over the effects diagonal in F with eigenvalue ratios near 1. □

4.3 Real, quaternionic and exceptional types

Theorem 4.5. *On $M_n(\mathbb{R})^{\mathrm{sa}}$ with $n \geq 2$, on $M_n(\mathbb{H})^{\mathrm{sa}}$ with $n \geq 3$, and on $H_3(\mathbb{O})$, every sequential product is the Lüders product.*

Proof. For rank at least three, every invertible effect is diagonal in some Jordan frame, so Proposition 4.4(i) gives $L_a = Q_{\sqrt{a}}$ on invertible effects, and Lemma 4.9 extends it to all effects. The real qubit $M_2(\mathbb{R})^{\mathrm{sa}} = V_2$ is the case $d = 2$ of Theorem 5.2. □

4.4 The complex type

The per-frame parameter of Proposition 4.4(ii) is globalized across frames. Two frames are *adjacent* if they differ inside one rank-two projection and share the remaining $n-2$ atoms. The comparison uses a complex structure on a cross-coherence space that does not depend on how a rank-two block splits into atoms.

Lemma 4.6 (Splitting-independent complex structure on a cross-coherence space)**.** *Let $J = M_n(\mathbb{C})^{\mathrm{sa}}$, $n \geq 3$, let q be a rank-two projection and p_k an atom with $q \circ p_k = 0$, and let $X = \{x : q \circ x = \frac{1}{2}x = p_k \circ x\}$ be the cross-coherence space, identified with the self-adjoint part of $qM_n(\mathbb{C})p_k \oplus p_kM_n(\mathbb{C})q$: each $x \in X$ is $x = z + z^*$ with corner $z = qxp_k$. Define the real-linear map*

$$\mathcal{J}_{q,k}(x) = iz - i\,(z^*), \qquad z = qxp_k.$$

Then:

(i) *$\mathcal{J}_{q,k}$ maps X to X with $\mathcal{J}_{q,k}^2 = -\mathbb{1}_X$, and depends only on q, p_k, and the fixed ambient complex scalar i of the labelled matrix algebra, not on any splitting $q = p_1 + p_2$ into atoms.*

(ii) *Every inner automorphism Ad_u (u unitary) with $uqu^* = q$ and $up_ku^* = p_k$ commutes with $\mathcal{J}_{q,k}$ on X.*

(iii) *Let $\{p_l\}_{l\geq 3}$ be pairwise-orthogonal atoms with $q + \sum_{l\geq 3} p_l = e$ and p_k among them, and $a = \lambda q + \sum_{l\geq 3} \lambda_l p_l$ with $\lambda, \lambda_l \in (0,1)$. Then for every $t \in \mathbb{R}$,*

$$\mathrm{Ad}_{a^{it}}\big|_X = \exp\big(t\,(\log\lambda - \log\lambda_k)\,\mathcal{J}_{q,k}\big).$$

Proof. The corner $z = qxp_k$ is determined by x. As $(iz^*)^* = -iz$, the element $\mathcal{J}_{q,k}(x)$ is self-adjoint and lies in $qM_n(\mathbb{C})p_k \oplus p_kM_n(\mathbb{C})q$, with corner iz; applying $\mathcal{J}_{q,k}$ again gives $-z - z^* = -x$. Only q, p_k and i enter, giving (i). For (ii), $q(uxu^*)p_k = u\,(qxp_k)\,u^*$, so $\mathcal{J}_{q,k}(\mathrm{Ad}_u\,x) = u(iz - iz^*)u^* = \mathrm{Ad}_u(\mathcal{J}_{q,k}x)$. For (iii), $a^{it} = e^{it\log\lambda}q + \sum_{l\geq 3} e^{it\log\lambda_l}p_l$ is unitary, and since $z = qzp_k$ the corner of $\mathrm{Ad}_{a^{it}}(x)$ is $e^{it\log\lambda}\,z\,e^{-it\log\lambda_k} = e^{i\varphi}z$ with $\varphi = t(\log\lambda - \log\lambda_k)$. On X, multiplying the corner by $e^{i\varphi}$ is $\cos\varphi\,\mathbb{1}_X + \sin\varphi\,\mathcal{J}_{q,k} = \exp(\varphi\,\mathcal{J}_{q,k})$. □

Lemma 4.7 (Adjacent-frame constancy)**.** *If F and F' are adjacent Jordan frames of $M_n(\mathbb{C})^{\mathrm{sa}}$, $n \geq 3$, then $t_F = t_{F'}$.*

Proof. Let $q = p_1 + p_2 = p_1' + p_2'$ be the rank-two projection inside which they differ, and fix a shared atom p_k, $k \geq 3$, with cross-coherence space X. Take $a = \lambda q + \sum_{l\geq 3}\lambda_l p_l$ with $\lambda, \lambda_l \in (0,1)$. This invertible effect is diagonal in both F and F', so Proposition 4.4(ii) gives $\mathrm{Ad}_{a^{it}{}_F} = \Theta_a = \mathrm{Ad}_{a^{it}{}_{F'}}$. Restricting to X with Lemma 4.6(iii), whose generator $\mathcal{J}_{q,k}$ is the same for both frames by Lemma 4.6(i),

$$\exp\big(t_F\,s\,\mathcal{J}_{q,k}\big) = \exp\big(t_{F'}\,s\,\mathcal{J}_{q,k}\big), \qquad s = \log\lambda - \log\lambda_k.$$

As λ varies, s ranges over an interval. Differentiating at an interior point and cancelling the invertible exponential gives $t_F\mathcal{J}_{q,k} = t_{F'}\mathcal{J}_{q,k}$, and $\mathcal{J}_{q,k} \neq 0$. So $t_F = t_{F'}$. □

Lemma 4.8 (Frame connectivity)**.** *For $n \geq 2$, the graph on Jordan frames of $M_n(\mathbb{C})^{\mathrm{sa}}$ whose edges join adjacent frames is connected.*

Proof. Frames are orthonormal bases up to phase and ordering. Let $p = ww^*$ be an atom of the target frame and write $w = \sum_i c_i v_i$ in the current orthonormal basis $\{v_i\}$. If $m > 1$ of the coefficients are nonzero, choose $i \neq j$ with $c_i, c_j \neq 0$, put $\rho = (|c_i|^2 + |c_j|^2)^{1/2} > 0$ and $u = \rho^{-1}(c_i v_i + c_j v_j)$, and complete u to an orthonormal pair $\{u, u^\perp\}$ of $\operatorname{span}\{v_i, v_j\}$. Replacing $\{v_i, v_j\}$ by $\{u, u^\perp\}$ is one edge of the graph. In the new basis w has component ρ along u and 0 along $u^\perp$, so the number of nonzero coordinates of w drops by one. After at most $n-1$ edges p is an atom of the current frame; all later moves may be taken inside $p^\perp$, and induction on n installs the remaining atoms, the base case $n = 2$ being a single move. □

Lemma 4.9 (Singular extension)**.** *Let $(\cdot)$ be a sequential product on J. If $L_a = Q_{\sqrt{a}}$ for every invertible effect a, then $L_a = Q_{\sqrt{a}}$ for every effect a. If $J = M_n(\mathbb{C})^{\mathrm{sa}}$ and $a \cdot b = a^{1/2+it}\, b\, a^{1/2-it}$ for every invertible effect a and every effect b, then the same holds for every effect a.*

Proof. For an effect a, $a_\varepsilon = (1-\varepsilon)a + \varepsilon e$ is invertible for $\varepsilon \in (0, 1]$ and $a_\varepsilon \to a$. (S2) gives $L_{a_\varepsilon} b \to L_a b$, and the functional calculus makes $Q_{\sqrt{a_\varepsilon}} b$ and $a_\varepsilon^{1/2+it}\, b\, a_\varepsilon^{1/2-it}$ continuous in ε up to $\varepsilon = 0$, with the value 0 at eigenvalue 0. □

Theorem 4.10 (Complex type)**.** *On $M_n(\mathbb{C})^{\mathrm{sa}}$, $n \geq 3$, there is a unique $t \in \mathbb{R}$ with $a \cdot b = a^{1/2+it}\, b\, a^{1/2-it}$ for all effects a, b.*

Proof. Each frame carries one parameter (Proposition 4.4), adjacent frames carry the same one (Lemma 4.7), and any two frames are joined by finitely many adjacent moves (Lemma 4.8), so a single t serves every frame. Every invertible effect is diagonal in some frame, and Lemma 4.9 extends the formula to all effects. Uniqueness is Proposition 4.4(ii). □

Proof of Theorem 2. Theorem 4.5 and Theorem 4.10, the four coordinate types exhausting the simple EJAs of rank at least three. □

Lemma 4.11 (Sufficiency of the twisted products)**.** *For every $n \geq 2$ and every $t \in \mathbb{R}$, the twisted product $\circ_t$ of (1) is a sequential product on $M_n(\mathbb{C})^{\mathrm{sa}}$.*

Proof. Set $g(\lambda) = \lambda^{1/2+it}$ with $g(0) := 0$, so that $|g(\lambda)|^2 = \lambda$ and $g(\lambda\mu) = g(\lambda)g(\mu)$ on $[0, \infty)$. Then $a \circ_t b = f_a(a)\, b\, \overline{f_a}(a)$ with $f_a = g|_{\mathrm{sp}(a)}$ is the functional-calculus product of this character family. Shen and Wu's Theorem 3.1 [17] states that $a \diamond b = f_a(a)\, b\, \overline{f_a}(a)$ is a sequential product on the effects of a finite-dimensional complex Hilbert space, in the sense of the sequential effect algebra axioms (SEA1)–(SEA5) of Gudder and Greechie [8], if and only if two conditions hold: (a) $|f_a(\lambda)| = \sqrt{\lambda}$ on the spectrum of a, and (b) for commuting a, b, $f_a(a) f_b(b) = \zeta\, f_{ab}(ab)$ for some unimodular ζ. Here (a) is $|g(\lambda)|^2 = \lambda$, and (b) holds with $\zeta = 1$ because commuting positive matrices are jointly diagonalizable and g is multiplicative. Those five axioms are exactly (S1) and (S3)–(S7), regrouped: (SEA1) is (S1); (SEA2) is (S3); (SEA3) is (S4); (SEA4) is (S5) together with the complement clause of (S6); and (SEA5) is (S7) together with the sum clause of (S6). Hence (S1) and (S3)–(S7) hold [17, Thm. 3.1, Ex. 3.1]; their Lemma 3.3 also shows that effects compatible for such a product commute, and the converse is immediate from (a), since $f_a(a)$ commutes with every b that commutes with a. For (S2), g is continuous on $[0, \infty)$, so $a \mapsto a^{1/2+it}$ is norm-continuous on the positive cone. □

Remark 4.12 (The labelled classification space). For a fixed finite-dimensional EJA J, $\mathcal{M}_{\mathrm{SP}}(J)$ is the set of sequential products on its fixed (labelled) effect algebra, before any quotient by Jordan automorphisms and with no topology imposed. For $M_n(\mathbb{C})^{\mathrm{sa}}$, $n \geq 3$, the map $t \mapsto \circ_t$ is a bijection $\mathbb{R} \to \mathcal{M}_{\mathrm{SP}}$: well-definedness is Lemma 4.11, surjectivity Theorem 4.10, and injectivity follows from Remark 6.3, whose 2×2 computation embeds in a coherence corner and separates every real t. A unitary conjugation transports $\circ_t$ to itself, and the transpose to $\circ_{-t}$; so quotienting by all Jordan automorphisms identifies t with $-t$, and $\circ_t$ is not isomorphic to the Lüders product for $t \neq 0$.

5 Rank Two: Spin Factors

This section classifies the sequential products on every spin factor V_d, $d \geq 2$ (Section 2.2). Write $\mathfrak{so}(d)$ for the real skew $d \times d$ matrices, acting on the vector part of V_d, and recall from the introduction that $\mathcal{A}_d$ is the set of continuous maps $A : S^{d-1} \to \mathfrak{so}(d)$ with $A(v)v = 0$ and $A(-v) = -A(v)$.

Definition 5.1 (Spin products)**.** For $A \in \mathcal{A}_d$ define $\star_A$ on the effects of V_d as follows, writing $L_a^A b = a \star_A b$:

(i) for a scalar effect $a = \mu e$, $L_a^A = \mu I$;

(ii) for an invertible nonscalar effect $a = \lambda_+ p_v + \lambda_- p_{-v}$ with axis v,

$$L_a^A = Q_{\sqrt{a}}\big(1 \oplus \exp(\ell\, A(v))\big), \qquad \ell = \log(\lambda_+/\lambda_-);$$

(iii) for a singular nonscalar effect $a = \lambda p_v$, $\lambda \in (0, 1]$, $L_a^A = Q_{\sqrt{a}}$, that is $L_a^A(\beta, y) = \lambda(\beta + v \cdot y)\, p_v$.

The exponential is never evaluated at a zero eigenvalue. Writing the same effect in the reversed frame, $a = \lambda_- p_{-v} + \lambda_+ p_v$, changes $v \mapsto -v$ and $\ell \mapsto -\ell$, and oddness of A leaves the formula unchanged. So (ii) may be applied in either ordering. $\star_0$ is the Lüders product.

Theorem 5.2 (Spin factors)**.** *For $d \geq 2$ the map $A \mapsto \star_A$ is a bijection from $\mathcal{A}_d$ onto the sequential products on V_d. For every such product, two nonscalar effects are compatible if and only if their axes are collinear, and scalar effects are compatible with every effect.*

The proof occupies Sections 5.1 and 5.2.

5.1 Necessity

Let $(\cdot)$ be a sequential product on V_d. Extend L to the interior of the cone: for $x > 0$ put $L_x := s\, L_{x/s}$ for any $s \geq \|x\|_e$, which is independent of s by Lemma 3.2, and put $\Theta_x := Q_{x^{-1/2}} L_x$. Then $L_{cx} = cL_x$ and $\Theta_{cx} = \Theta_x$ for $c > 0$, and on invertible effects Θ_x is the comparison automorphism of Proposition 3.7.

Step 1: the frame group. Let $x = \lambda_+ p_v + \lambda_- p_{-v} > 0$. Normalizing to an invertible effect, Proposition 3.7 and its fixing clause make Θ_x a Jordan automorphism fixing p_v and p_{-v}. So $\Theta_x = 1 \oplus O_x$ with $O_x \in \mathrm{O}(d)$ and $O_x v = v$ (Section 2.2). For x, x' positive and diagonal in the frame $\{p_v, p_{-v}\}$, normalizing each into the effects and using Proposition 3.7(iii) with scale

invariance gives $\Theta_{x\circ x'} = \Theta_x \Theta_{x'}$. Also $\Theta_{ce} = I$ by Lemma 3.2. Since $x = \sqrt{\lambda_+\lambda_-}\,(e^{\ell/2}p_v + e^{-\ell/2}p_{-v})$ with $\ell = \log(\lambda_+/\lambda_-)$, Θ_x depends only on (v,ℓ), and $\ell \mapsto O_v(\ell) := O_x|_{v^\perp}$ is a continuous homomorphism $(\mathbb{R},+) \to \mathrm{SO}(v^\perp)$, continuity coming from Proposition 3.7(ii). By the theorem on continuous one-parameter matrix groups [11, Ch. 2] there is a unique skew $A(v)$ on $v^\perp$ with $O_v(\ell) = \exp(\ell A(v))$. Extend $A(v)$ by $A(v)v = 0$.

Step 2: oddness. The same x has frame data $(-v,-\ell)$ in the reversed ordering, so $\exp(\ell A(v)) = \exp(-\ell A(-v))$ for all ℓ, and uniqueness of the generator gives $A(-v) = -A(v)$.

Step 3: boundedness. Suppose $\|A(v_j)\| \to \infty$ in the Euclidean operator norm. Put $s_j = \pi/\|A(v_j)\|$ and, for a fixed $\mu \in (0,1)$, $a_j = \mu(e^{s_j/2}p_{v_j} + e^{-s_j/2}p_{-v_j})$. Eventually these are invertible effects, and $a_j \to \mu e$ whatever the v_j. On a maximal-frequency rotation plane of $A(v_j)$, $\exp(s_j A(v_j))$ acts as $-I$, so $\|\Theta_{a_j} - I\| \geq 2$, contradicting continuity of Θ at μe, where $\Theta_{\mu e} = I$. Hence $M := \sup_v \|A(v)\| < \infty$.

Step 4: continuity. If $M = 0$ there is nothing to prove. Otherwise fix $s > 0$ with $sM < \pi$ and $\mu e^{s/2} < 1$. Then $v \mapsto \Theta_{\mu(e^{s/2}p_v + e^{-s/2}p_{-v})}$ is continuous, and its spatial part $\exp(sA(v))$ has all rotation angles in $(-\pi,\pi)$. On that set the principal matrix logarithm is continuous and inverts exp on skew matrices, so $A(v) = s^{-1}\operatorname{Log}\exp(sA(v))$ is continuous. No continuous choice of basis of $v^\perp$ is needed. So $A \in \mathcal{A}_d$.

Step 5: the product is $\star_A$. For invertible nonscalar a, $L_a = Q_{\sqrt{a}}\Theta_a = L_a^A$ by Step 1. For scalar $a = \mu e$, $L_a = \mu I$ by Lemma 3.2. For singular nonscalar $a = \lambda p_v$, let $a_\varepsilon = \lambda p_v + \varepsilon p_{-v}$ with $0 < \varepsilon < \lambda$. Then $L_{a_\varepsilon} = Q_{\sqrt{a_\varepsilon}}(1 \oplus \exp(\log(\lambda/\varepsilon)A(v)))$ has multipliers λ, ε and $\sqrt{\lambda\varepsilon}\exp(\cdots)$ on $\mathbb{R}p_v$, $\mathbb{R}p_{-v}$ and $v^\perp$. The rotation is orthogonal, hence bounded, so (S2) gives $L_a = \lim_{\varepsilon\to 0} L_{a_\varepsilon} = \lambda Q_{p_v} = Q_{\sqrt{a}}$. Finally A is determined by the product, being the generator in Step 1.

5.2 Sufficiency

Fix $A \in \mathcal{A}_d$. Each L_a^A is a composition of $Q_{\sqrt{a}}$ with an orthogonal map $1 \oplus R$ that preserves the cone, so it is positive, linear, and satisfies $L_a^A e = a$; hence L_a^A maps effects into $[0,a]$, and (S1) and (S3) hold. Also $L_{ca}^A = cL_a^A$ for $c \in [0,1]$, directly from the definition.

An explicit formula. Write $a = (\alpha, ru)$ and $b = (\beta, sv)$ with $r,s \geq 0$ and unit vectors u,v, and put $k_a = \sqrt{\alpha^2 - r^2}$, the geometric mean of the eigenvalues of a. Let $R_a = \exp(\log(\lambda_+/\lambda_-)A(u))$ for invertible nonscalar a and $R_a = I$ otherwise; when a is singular its transverse multiplier k_a is 0. Decomposing b along $p_{\pm u}$ and $u^\perp$ gives

$$(a \star_A b)_{\text{scalar}} = \alpha\beta + rs\,(u\cdot v), \qquad (a \star_A b)_{\text{vector}} = \beta r\,u + \alpha s\,(u\cdot v)\,u + k_a s\,R_a\big(v - (u\cdot v)u\big). \quad (7)$$

(S4). Nonzero effects have $\alpha,\beta > 0$. If $a \star_A b = 0$ then $\alpha\beta = -rs(u\cdot v) \leq rs \leq \alpha\beta$, forcing $\alpha = r$, $\beta = s$ and $u\cdot v = -1$. So a and b lie on opposite primitive rays, where the singular formula gives $b \star_A a = 0$. If either factor is 0 both products vanish.

Exact compatibility. Let $r,s > 0$, $z = u\cdot v$ and $w = u - zv$. Since R_a fixes u and preserves $u^\perp$, the u-component of $(a \star_A b)_{\text{vector}}$ is $\beta r + \alpha s z$. Since R_b fixes v and preserves $v^\perp \ni w$, the u-component of $(b \star_A a)_{\text{vector}}$ is $\alpha s z + \beta r z^2 + k_b r\langle w, R_b w\rangle$. If the products agree, then $\beta(1-z^2) = k_b\langle w, R_b w\rangle \leq k_b\|w\|^2 = k_b(1-z^2)$. As $s > 0$ gives $\beta > k_b$, this forces $z^2 = 1$. Conversely effects with collinear axes are diagonal in one frame and multiply spectrally, and scalar effects commute with all effects. This proves the compatibility statement of Theorem 5.2, singular effects included.

(S5)–(S7). If $a = \mu e$ is scalar, then $a \star_A (b \star_A c) = \mu\, b \star_A c = (\mu b) \star_A c = (a \star_A b) \star_A c$ by $L^A_{\mu b} = \mu L^A_b$; if $b = \nu e$ is scalar, both sides of (S5) equal $\nu\, a \star_A c$ by linearity of L^A_a and $L^A_{\nu a} = \nu L^A_a$. If a, b are compatible and nonscalar they share a frame $\{p_v, p_{-v}\}$. For strictly positive a, b the diagonal multipliers multiply, the transverse prefactors multiply, and the rotations $\exp(\ell_a A(v))$ and $\exp(\ell_b A(v))$ compose to $\exp((\ell_a + \ell_b)A(v))$, the log-ratios being taken in the same orientation. So $L^A_a L^A_b = L^A_{a \circ b}$, including when $a \circ b$ is scalar, where $\ell_a + \ell_b = 0$. If either factor is singular, its transverse multiplier vanishes and the diagonal formula still composes. Since $a \star_A b = a \circ b$ for compatible pairs, (S5) follows. For nonscalar a the effects compatible with a are exactly the effects in $\operatorname{span}\{p_v, p_{-v}\}$, a set closed under complements, admissible sums and $\star_A$; this gives (S6) and (S7).

(S2). We show $a \mapsto L^A_a$ is operator-norm continuous on all effects. Away from scalar and singular effects the axis, the eigenvalues and A are continuous. Near μe with $\mu > 0$, the log-ratio tends to 0 uniformly in the axis and $\|A\| \le M$, so $\exp(\ell A(v)) \to I$ and $Q_{\sqrt{a}} \to \mu I$; this includes $\mu = 1$, with limits within the effects. Near a singular λp_v with $\lambda > 0$, the axis tends to v, the diagonal multipliers converge, and the transverse multiplier $\sqrt{\lambda_+ \lambda_-}$ tends to 0 while the rotation stays of norm one. At 0, $\|L^A_a\|_{e \to e} = \|a\|_e \to 0$. This completes the proof of Theorem 5.2.

5.3 Consequences

Corollary 5.3 (The real qubit and the complex qubit)**.** (i) $\mathcal{A}_2 = \{0\}$*: every sequential product on* $V_2 \cong M_2(\mathbb{R})^{\mathrm{sa}}$ *is Lüders.*

(ii) $\mathcal{A}_3 \cong C(\mathbb{RP}^2, \mathbb{R})$ *through* $A(v) = -t([v])\, J_v$ *with* $J_v w = v \times w$*. Under* $V_3 \cong M_2(\mathbb{C})^{\mathrm{sa}}$ *the product* $\star_A$ *is*

$$a \cdot b = a^{1/2 + i\, t(\mathrm{fr}\, a)}\; b\; a^{1/2 - i\, t(\mathrm{fr}\, a)} \qquad (a \text{ nonscalar}),$$

with $\mathrm{fr}\, a \in \mathbb{RP}^2$ *the frame of* a*, and the Lüders product for scalar* a*. So the sequential products on* $M_2(\mathbb{C})^{\mathrm{sa}}$ *are in bijection with* $C(\mathbb{RP}^2, \mathbb{R})$*, and the constant functions give Liu and Wu's family* (1)*.*

Proof. (i) A skew map of $\mathbb{R}^2$ annihilating a nonzero vector is 0. (ii) An element of $\mathfrak{so}(3)$ killing v is a multiple of J_v, so $A(v) = -t(v) J_v$ for a continuous t; since $J_{-v} = -J_v$, oddness of A is evenness of t, that is, t descends to $\mathbb{RP}^2$. For $a = \lambda_+ P_v + \lambda_- P_{-v}$ invertible, $a^{it} = (\lambda_+ \lambda_-)^{it/2} \exp(\frac{i}{2} t \ell\, v \cdot \sigma)$ with $\ell = \log(\lambda_+ / \lambda_-)$, and $\frac{d}{d\ell} \operatorname{Ad}_{a^{it}}(w \cdot \sigma)\big|_{\ell=0} = \frac{it}{2}[v \cdot \sigma, w \cdot \sigma] = -t\,(v \times w) \cdot \sigma$. So $\operatorname{Ad}_{a^{it}}$ fixes I and $v \cdot \sigma$ and acts on $v^\perp$ as $\exp(-t\ell J_v)$, and $a^{1/2+it}\, b\, a^{1/2-it} = Q_{\sqrt{a}} \operatorname{Ad}_{a^{it}} b$ is $L^A_a b$ with $A(v) = -t J_v$. At singular and scalar a both sides are the Lüders value. □

The constructive half of (ii) specializes Shen and Wu's frame-dependent construction [17, Thm. 3.1, Ex. 3.2], whose frame parameter is arbitrary and need not satisfy (S2); the necessity half is new.

Example 5.4 (Frame dependence)**.** Fix a unit vector $\rho \in \mathbb{R}^3$ and let $t(v) = (v \cdot \rho)^2$. The product of Corollary 5.3(ii) is not of the form $\Phi^{-1}(\Phi(a) \circ_{t_0} \Phi(b))$ for any Jordan automorphism Φ and constant t_0. Indeed the Jordan automorphisms act on V_3 as $1 \oplus \mathrm{O}(3)$, and transport a constant twist t_0 to the constant $\pm t_0$, while t takes both values 0 and 1.

Remark 5.5 (Higher spin factors). For every $d \geq 3$ there are nonzero fields: for a nonzero alternating 3-form ω on $\mathbb{R}^d$, $\langle x, A(v)y\rangle = \omega(v, x, y)$ defines $A \in \mathcal{A}_d$. So Theorem 5.2 is not a rigidity statement for any $d \geq 3$, including $V_5 \cong M_2(\mathbb{H})^{\mathrm{sa}}$ and $V_9 \cong H_2(\mathbb{O})$, in contrast with the rank-at-least-three quaternionic and exceptional algebras. A Jordan automorphism $1 \oplus O$ transports $\star_A$ to $\star_{A'}$ with $A'(v) = OA(O^{-1}v)O^{-1}$. If a product is covariant under $\mathrm{SO}(d)$, then $A(v)$ is invariant under the stabilizer $\mathrm{SO}(v^\perp) \cong \mathrm{SO}(d-1)$ acting on $\mathfrak{so}(v^\perp)$. For $d \geq 4$, $\mathfrak{so}(d-1)$ has trivial centre, so $A = 0$ and the product is Lüders. For $d = 3$, $\mathfrak{so}(2)$ is abelian, and covariance leaves exactly the constant functions t, the Liu–Wu line.

5.4 Proof of the classification

Proof of Theorem 1. Proposition 3.6 decomposes the product over the simple summands, and its converse assembles independent summand products. On the rank-one summand $\mathbb{R}$, L_a is positive linear with $L_a(1) = a$, so $a \cdot b = ab$. The summands of rank at least three are Theorem 2, and the rank-two summands are the spin factors, classified by Theorem 5.2. For the converse, each listed value is attained: the Lüders product is a sequential product on every EJA (van de Wetering [18, Thm. A.6] for (S1) and (S3)–(S7), with (S2) from continuity of the functional calculus); every $t \in \mathbb{R}$ is attained by Lemma 4.11; and every $A \in \mathcal{A}_d$ by Theorem 5.2. Uniqueness of the parameters is Remark 4.12 and Theorem 5.2. □

6 The Physical Content of the Freedom

6.1 Dynamical correspondences

Alfsen and Shultz [2, 1, 3] define a *dynamical correspondence* on a unital JB-algebra A as a linear map $a \mapsto \psi_a$ into the skew order derivations (the Jordan derivations) of A satisfying $[\psi_a, \psi_b] = -[T_a, T_b]$ and $\psi_a(a) = 0$. The first condition is a quantization rule relating the commutator of generators to the commutator of Jordan multipliers; the second says that the motion generated by an observable fixes that observable. Their Theorem 1 [2], proved in [1], states that a unital JB-algebra is the self-adjoint part of a C*-algebra if and only if it admits a dynamical correspondence, and that dynamical correspondences correspond to the Jordan-compatible C*-products through $\psi_a b = \frac{i}{2}(ab - ba)$.

Proof of Theorem 3. (ii) On $M_n(\mathbb{C})^{\mathrm{sa}}$ put $\psi_h(x) = \frac{i}{2}(hx - xh)$. It is a derivation of the Jordan product, $\psi_h(h) = 0$, and the Jacobi identity gives $[\psi_a, \psi_b]x = -\frac{1}{4}[[a, b], x] = -[T_a, T_b]x$; so ψ is a dynamical correspondence, the one of the matrix product. For invertible a, $\exp(2t\psi_{\log a})x = e^{it\log a}\, x\, e^{-it\log a} = \mathrm{Ad}_{a^{it}}\, x$, so $Q_{\sqrt{a}} \exp(2t\psi_{\log a})b = a^{1/2}a^{it}\, b\, a^{-it}a^{1/2} = a \circ_t b$. Theorem 4.10 and Lemma 4.11 complete (ii).

(i) If $J = M_n(\mathbb{C})^{\mathrm{sa}}$, it is the self-adjoint part of $M_n(\mathbb{C})$ and carries $\circ_t$ with $t \neq 0$, which is not Lüders (Remark 6.3). Otherwise J is of real, quaternionic or exceptional type, and every product is Lüders by Theorem 4.5. Such a J is not the self-adjoint part of a C*-algebra: a finite-dimensional C*-algebra is a direct sum of full matrix algebras $M_k(\mathbb{C})$, so a simple self-adjoint part is some $M_k(\mathbb{C})^{\mathrm{sa}}$, whose Peirce multiplicity is 2, whereas J has multiplicity 1, 4 or 8. By Alfsen and Shultz's theorem, J admits no dynamical correspondence. □

Remark 6.1 (Rank two). On $V_3 \cong M_2(\mathbb{C})^{\mathrm{sa}}$, $\psi_{v\cdot\sigma}$ acts on the traceless part as $w \cdot \sigma \mapsto \frac{i}{2}[v\cdot\sigma, w\cdot\sigma] = -(v\times w)\cdot\sigma$, that is, as $-J_v$. So the field of Corollary 5.3 is $A(v) = t([v])\,\psi_{v\cdot\sigma}$, and the qubit's products are frame-dependent multiples of the dynamical correspondence. For $d \neq 3$ the spin factor V_d ($d \geq 2$) is not the self-adjoint part of a C*-algebra: a simple self-adjoint part of a finite-dimensional C*-algebra is some $M_k(\mathbb{C})^{\mathrm{sa}}$, of rank k, and rank two forces $k = 2$, that is V_3. So V_d admits no dynamical correspondence. Nonetheless V_d carries non-Lüders products for every $d \geq 4$ (Remark 5.5). On $M_n(\mathbb{C})^{\mathrm{sa}}$ with $n \geq 3$, overlapping frames force the per-frame generators to agree (Lemma 4.7), and the resulting global generator is a multiple of the dynamical correspondence (Theorem 3(ii)); in rank two distinct frames share no atom.

6.2 Sharp measurements and detection

Corollary 6.2 (Sharp first effects)**.** *For every sequential product on a finite-dimensional EJA, every idempotent p and every effect b, $p \cdot b = Q_p\, b$.*

Proof. By Proposition 3.6 it suffices to treat simple J. On the Lüders types there is nothing to prove. On $M_n(\mathbb{C})^{\mathrm{sa}}$, $n \geq 3$, the support-logarithm convention gives $p^{1/2\pm it} = p$, so $p \circ_t b = pbp = Q_p b$. On a spin factor the idempotents are 0, e and the p_v, for which Definition 5.1(i) and (iii) give Q_p. On $\mathbb{R}$ it is immediate. □

On Hilbert-space effects this is Shen and Wu's Theorem 2.1 [17], proved there for every sequential-effect-algebra product, without (S2); Corollary 6.2 extends it to every finite-dimensional Euclidean Jordan algebra. So every departure from the Lüders rule occurs in unsharp measurements.

Remark 6.3 (Operational detectability of the twist). The parameter t in $a \circ_t b = a^{1/2+it}\, b\, a^{1/2-it}$ has an interferometric signature on a single qubit, and embeds in any coherence corner of $M_n(\mathbb{C})^{\mathrm{sa}}$. Take the unsharp first effect $a_r = \mathrm{diag}(e^{-r}, 1)$, the sharp second effect $b = |+\rangle\langle+|$, and the state $\rho_\phi = \frac{1}{2}\left(\begin{smallmatrix} 1 & e^{-i\phi} \\ e^{i\phi} & 1 \end{smallmatrix}\right)$. The probability that a_r and then b both return "yes" is

$$p_t(r,\phi) = \mathrm{tr}\big(\rho_\phi\,(a_r \circ_t b)\big) = \frac{1+e^{-r}}{4} + \frac{e^{-r/2}}{2}\,\cos(\phi - tr).$$

Operationally: prepare $|\psi_\phi\rangle = (|0\rangle + e^{i\phi}|1\rangle)/\sqrt{2}$, so that $\rho_\phi = |\psi_\phi\rangle\langle\psi_\phi|$, with a phase gate setting ϕ; perform the unsharp two-outcome measurement of a_r, whose strength sets r; then measure in the basis $|\pm\rangle$. Reading $a \cdot b$ as "a, then b," $p_t(r,\phi)$ is the relative frequency of the outcome pair (yes, +). Scanning the state phase isolates two quadratures, $p_t(r,0) \propto \cos(tr)$ and $p_t(r,\frac{\pi}{2}) \propto \sin(tr)$ above the t-independent pedestal $\frac{1}{4}(1+e^{-r})$. A single unsharp $r > 0$ fixes t modulo $2\pi/r$, and two values with irrational ratio fix t in the noiseless model. Unsharpness is essential, by Corollary 6.2. Liu and Wu already compute the qubit form of their product and read t as a physical "time parameter" [14, Cor. 4.4, Rem. 1]; the state-dependent detection formula is ours.

Remark 6.4 (Composite systems). Let $A = M_n(\mathbb{C})^{\mathrm{sa}}$, $B = M_m(\mathbb{C})^{\mathrm{sa}}$ with $n, m \geq 2$, and $AB = M_{nm}(\mathbb{C})^{\mathrm{sa}}$, so that AB has rank at least four. Say that the product of AB *restricts to* that of A if $(a \otimes \mathbb{1}) \cdot (b \otimes \mathbb{1}) = (a \cdot b) \otimes \mathbb{1}$ for all effects a, b of A, and similarly for B. By

Theorem 4.10, AB carries $\circ_t$ for one t, and $(a \otimes \mathbb{1})^{1/2+it}(b \otimes \mathbb{1})(a \otimes \mathbb{1})^{1/2-it} = (a \circ_t b) \otimes \mathbb{1}$. So if the product of AB restricts to that of A, the product of A is $\circ_t$ with the composite's t; if it also restricts to that of B, the same holds for B, and A, B and AB share one twist. In particular a qubit $A = M_2(\mathbb{C})^{\mathrm{sa}}$ that is a restricting factor of one such composite has a constant frame function (Corollary 5.3). For a family of complex systems in which every pair has a composite restricting to both factors, all members share one twist. Fiorentino and Weigert [6] single out the analogous compatibility of local and global updates as the condition that extends the single-system Lüders rule to composites.

6.3 Selectors

Several classical side conditions select $t = 0$ on the complex type. The last two are due, in greater generality, to van de Wetering and to Gudder and Latrémolière (Section 7). Write $E(x, y)$ for the action of $a \cdot (\cdot)$ on a coherence block V_{ij} when a has eigenvalues x, y on the atoms p_i, p_j; by Theorem 4.10 it is $E(x, y) = \sqrt{xy}\,\exp(t \log(x/y)\,\mathcal{J})$ with $\mathcal{J}$ the block's complex structure. *Peirce exchange covariance* means $E(x, y) = E(y, x)$ with $\mathcal{J}$ held fixed.

Corollary 6.5 (Selectors that fix $t = 0$)**.** *On $M_n(\mathbb{C})^{\mathrm{sa}}$, $n \geq 3$, a sequential product is the Lüders product if any of the following holds: (i) Peirce exchange covariance on every coherence block; (ii) trace-form symmetry, $\langle a \cdot b, c\rangle = \langle b, a \cdot c\rangle$; (iii) covariance under every unital order automorphism (the transpose suffices).*

Proof. (i) $E(x, y) = E(y, x)$ for all x, y iff $t = 0$. (ii) Peirce spaces are trace-orthogonal, so symmetry is self-adjointness of each block rotation $\exp(t \log(x/y)\,\mathcal{J})$; a real plane rotation is self-adjoint only at angles in $\pi\mathbb{Z}$, but $t \log(x/y)$ ranges over an interval, so $t = 0$. (iii) The transpose acts on each block by $\mathcal{J} \mapsto -\mathcal{J}$, forcing $t = -t$. □

The twisted family shows that three weaker conditions do not select $t = 0$. The twisted products have symmetric atomic transition probabilities $\mathrm{tr}(pq)$, so transition-probability symmetry does not select. Unitary rotations commute with $t\mathcal{J}$, so covariance under the identity component does not select; only the orientation-reversing transpose kills t. The twisted product of invertibles is invertible, so closure of the invertibles does not select, while van de Wetering's strictly stronger invertibility preservation [18, Def. 5.16], which adds an inverse law relating the product of two invertible effects to the product of their inverses (the latter taken on the positive cone, as in his definition), does select given norm-continuity [18, Thm. 5.19].

7 Relation to Prior Work

Uniqueness with an added condition. Gudder and Latrémolière [9] characterize the Lüders product on Hilbert-space effects by five conditions, one a trace duality, the Hilbert-space form of selector (ii) of Corollary 6.5. The twisted products satisfy their other four conditions, including the weak associativity $A \cdot (A \cdot B) = A^2 \cdot B$. Van de Wetering [18] proves three uniqueness theorems on Euclidean Jordan algebras, each adding one condition to the axioms: invariance of the product under order automorphisms with norm-continuity

(his Theorems 5.9 and 5.11), inner-product symmetry (Theorem 5.14), and invertibility preservation (Theorem 5.19). His Section 5 introduces the comparison map used here and, on the complex Hilbert-space type and for simple invertible a, reduces it to conjugation by a unitary u_a in the bicommutant of a (his Lemma 5.8). Theorem 2 determines the dependence $a \mapsto u_a$ from the axioms alone: in finite dimension and rank at least three, $u_a = a^{it}$ up to a phase, for one global t. His thesis records that the fully general instrument form of an update is not compatible with the sequential axioms, the reason being "still not entirely clear" [20, Rem. 2.2.7].

The twisted family. The family (1) is Liu and Wu's [14, Thm. 4.3, Cor. 4.4, Rem. 1], an existence result that answers Gudder's uniqueness question negatively. Shen and Wu [17, Thm. 3.1] characterize which functional-calculus families $a \mapsto f_a(a)\, b\, f_a(a)^*$ are products under their axioms, which do not include (S2); their Example 3.2 allows an arbitrary parameter per qubit frame. Liu, Wu and Wu [15, Thm. 2] observe that each such product value is $U(a \bullet b)U^*$ for a unitary U commuting with a and depending on a, and call the products unitarily equivalent to Lüders'. That is the pointwise shape (2), not an isomorphism of products: for $t \neq 0$, $\circ_t$ is not carried to the Lüders product by any Jordan automorphism (Remark 4.12), and the two are operationally distinguishable (Remark 6.3). None of these works shows that every product satisfying the axioms has the functional-calculus form; that is the necessity half of Theorem 1.

Structure and neighbouring problems. Theorem 2.2 is van de Wetering's structure theorem. Westerbaan, Westerbaan and van de Wetering classify the normal sequential effect algebras into three structural types [22]; the present problem fixes the effect algebra and classifies the products it carries. Jenčová and Pulmannová study spectrality in convex sequential effect algebras [12]; in finite dimension the spectral theorem supplies what is needed here. Gudder studies the operation-relative sequential product $a[\mathcal{I}]b$, whose value depends on the operation $\mathcal{I}$ chosen to measure a [10]; the products classified here are effect-level, and on the complex type each is realized by the single-Kraus operation $\rho \mapsto (U_a^t)^* \sqrt{a}\, \rho\, \sqrt{a}\, U_a^t$. Niestegge's rank-two generalized spin factors over ℓ^p with asymmetric transition probabilities [16, Thm. 7.1, §10] lie outside the Euclidean Jordan setting and outside this classification.

Alternatives to the projection postulate. Fiorentino and Weigert [6] study state-update rules that Nature could have chosen instead of Lüders' for projective measurements, build consistent foil theories from them, and identify operational properties that single out the projection postulate, with particular attention to composite systems. Their rules act on states and need not come from an effect-level product. Within the effect-level axioms and in finite dimension, no alternative survives on sharp measurements (Corollary 6.2; on Hilbert space, Shen and Wu [17, Thm. 2.1]), and on unsharp ones the alternatives are the products classified here.

Dynamical correspondences. Alfsen and Shultz [2, 1, 3] introduced dynamical correspondences to characterize the self-adjoint parts of C*-algebras among JB-algebras, and connected them to Connes' notion of orientation. We are not aware of a previous connection

between dynamical correspondences and sequential products. Theorem 3 shows that in rank at least three the freedom left by the sequential axioms is exactly the real line spanned by a dynamical correspondence.

8 Closing Remarks

In finite dimension the classification is complete. Two questions remain. First, infinite dimensions: the argument uses finite spectral resolutions and Lie theory on a finite-dimensional stabilizer, and extending it to type I factors or JBW-algebras would require new continuity arguments. Second, the physical value of t: the axioms leave it free, the natural symmetry and duality conditions of Corollary 6.5 set it to 0, and Remark 6.3 gives an experiment that could bound it for any concrete measurement scheme built to realize a sequential product.

Acknowledgments

The author thanks John van de Wetering for making his sequential product theorems publicly available and clearly stated.

Claude models (Anthropic; Opus 4.8, Opus 5.5, Fable 5 and Fable 5.1) were used for literature search, exploratory mathematical discussion, drafting, proof review, and the Lean 4 development. GPT models (OpenAI; `gpt-5.6-sol` and `gpt-6-sol`) were used to check arguments and review drafts. The author is responsible for the content.

Author Declarations

Conflict of Interest

The author has no conflicts to disclose.

Author Contributions

Bryan Ehrlich: Conceptualization (lead); Formal analysis (lead); Investigation (lead); Methodology (lead); Software (lead); Validation (lead); Writing, original draft (lead); Writing, review and editing (lead).

Data Availability

The Lean 4 development is openly available in two public GitHub repositories. The curated release `ehrlich-b/sequential-products-lean`, at the immutable commit `9c6d92f5fc222d9d1646ae6044dcec053900ca83`, contains the real and complex classifications (`Necessity.real_classification` and `Necessity.complex_classification_unconditional`), the sufficiency of the twisted products (`HermitianMat.twistSequentialProduct`), and the complex-qubit classification (`RankTwo.qubit_classification_up_to_effects`). Its statement files for the two classification

theorems are registered in the Palomar registry of Lean-verified mathematics as entry `PALOMAR-2026-08-22-000006`. The development repository `ehrlich-b/research`, directory `twist-normal-form-lean`, at the immutable commit `3fffbd9663cd7b63def09aa5131d5df0784a4ceb`, additionally contains the central-imaginary argument of Section 4 on the quaternionic and octonionic carriers (`EJA.quaternionic_luders` and `EJA.albert_luders`) and the Lüders product as a sequential product at the generality of an abstract Euclidean Jordan algebra (`EJA.ludersSequentialProduct`, with first-argument continuity proved in `EJA.ludersSequentialProduct_firstArgContinuous`). Both are self-contained Lake projects built with Lean 4 against Mathlib, with the Mathlib revision pinned in `lake-manifest.json`. Each includes a separately invoked axiom audit, `AxiomAudit.lean`, requiring every persisted declaration in its tracked `RadicalRelativity` source tree to depend only on Lean's three core axioms `propext`, `Classical.choice` and `Quot.sound`. No other data were generated or analysed in this work.

References

[1] Erik M. Alfsen and Frederic W. Shultz. On orientation and dynamics in operator algebras. Part I. *Communications in Mathematical Physics*, 194:87–108, 1998.

[2] Erik M. Alfsen and Frederic W. Shultz. Orientation in operator algebras. *Proceedings of the National Academy of Sciences of the USA*, 95(12):6596–6601, 1998.

[3] Erik M. Alfsen and Frederic W. Shultz. *Geometry of State Spaces of Operator Algebras*. Birkhäuser, Boston, 2003.

[4] Howard Barnum, Matthew A. Graydon, and Alexander Wilce. Composites and categories of Euclidean Jordan algebras. *Quantum*, 4:359, 2020. arXiv:1606.09331.

[5] Jacques Faraut and Adam Korányi. *Analysis on Symmetric Cones*. Oxford University Press, 1994.

[6] Vincenzo Fiorentino and Stefan Weigert. Beyond the projection postulate and back: Quantum theories with generalised state-update rules. *Physical Review A*, 113:012204, 2026. arXiv:2506.06207.

[7] Stan Gudder. Open problems for sequential effect algebras. *International Journal of Theoretical Physics*, 44(12):2199–2206, 2005.

[8] Stan Gudder and Richard Greechie. Sequential products on effect algebras. *Reports on Mathematical Physics*, 49(1):87–111, 2002.

[9] Stan Gudder and Frédéric Latrémolière. Characterization of the sequential product on quantum effects. *Journal of Mathematical Physics*, 49(5):052106, 2008. arXiv:0803.3867.

[10] Stanley Gudder. Properties of sequential products, 2023. arXiv:2307.16327.

[11] Brian C. Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*, volume 222 of *Graduate Texts in Mathematics*. Springer, second edition, 2015.

[12] Anna Jenčová and Sylvia Pulmannová. Spectrality in convex sequential effect algebras. *International Journal of Theoretical Physics*, 62(8):193, 2023.

[13] Pascual Jordan, John von Neumann, and Eugene Wigner. On an algebraic generalization of the quantum mechanical formalism. *Annals of Mathematics*, 35(1):29–64, 1934.

[14] Weihua Liu and Junde Wu. The uniqueness problem of sequence product on operator effect algebra $\mathcal{E}(H)$. *Journal of Physics A: Mathematical and Theoretical*, 42(18):185206, 2009. arXiv:0812.0630.

[15] Weihua Liu, Zhaoqi Wu, and Junde Wu. Remarks on the sequential products, 2010. arXiv:1009.2593.

[16] Gerd Niestegge. Non-symmetric transition probability in generalized qubit models. *Foundations of Physics*, 54(1):9, 2024. arXiv:2208.07135.

[17] Jun Shen and Junde Wu. Sequential product on standard effect algebra $\mathcal{E}(H)$. *Journal of Physics A: Mathematical and Theoretical*, 42(34):345203, 2009. arXiv:0905.0596.

[18] John van de Wetering. Three characterisations of the sequential product. *Journal of Mathematical Physics*, 59(8):082202, 2018. arXiv:1803.08453.

[19] John van de Wetering. Sequential product spaces are Jordan algebras. *Journal of Mathematical Physics*, 60(6):062201, 2019. arXiv:1803.11139.

[20] John van de Wetering. *Quantum Theory from Principles, Quantum Software from Diagrams*. PhD thesis, Radboud University Nijmegen, 2021. arXiv:2101.03608.

[21] Hendrik van Imhoff and Mark Roelands. Order isomorphisms between cones of JB-algebras. *Studia Mathematica*, 254:179–198, 2020. arXiv:1904.09278.

[22] Abraham Westerbaan, Bas Westerbaan, and John van de Wetering. The three types of normal sequential effect algebras. *Quantum*, 4:378, 2020. arXiv:2004.12749.