

Sequential Products on Euclidean Jordan Algebras: Classification in Rank at Least Three and the Complex Qubit

Bryan Ehrlich

Independent researcher, Trumbull, Connecticut 06611, USA

ehrich.bryan@gmail.com

Abstract

A sequential product takes an ordered pair of effects (a, b) to a new effect $a \cdot b$, read as “measure a , then b ,” subject to seven standard axioms, one of them norm-continuity in the first argument. We classify these products on every finite-dimensional Euclidean Jordan algebra whose rank-two summands are real or complex qubits. The classification is typewise, because a sequential product cannot couple distinct simple summands. On real $M_n(\mathbb{R})^{\text{sa}}$ ($n \geq 2$), quaternionic $M_n(\mathbb{H})^{\text{sa}}$ ($n \geq 3$), exceptional $H_3(\mathbb{O})$, and rank-one summands, the Lüders product $a \bullet b = Q_{\sqrt{a}} b$ is forced; here Q is the quadratic representation, so on the matrix types this reads $\sqrt{a} b \sqrt{a}$. On each complex summand $M_n(\mathbb{C})^{\text{sa}}$ ($n \geq 3$) the products are instead exactly Liu–Wu’s twisted family $a \circ_t b = a^{1/2+it} b a^{1/2-it}$, one for each real t . Its members agree with Lüders on sharp first effects and on commuting pairs, and differ from it by an outcome-conditioned unitary back-action, detectable interferometrically. The complex qubit $M_2(\mathbb{C})^{\text{sa}}$ carries a function space $C(\mathbb{RP}^2, \mathbb{R})$ of products, in place of the single parameter that survives from rank three on. The remaining rank-two spin factors are left open. Adding trace-form symmetry to these axioms forces the Lüders product on every simple type of rank ≥ 3 , and we compute the freedom such selectors remove. That selector is van de Wetering’s inner-product symmetry, and the Jordan form of Gudder–Latréolière’s trace duality. The proof runs through van de Wetering’s comparison map and a frame-stabilizer argument on each type.

1 Introduction

How far the operational axioms of Definition 2.1 determine a sequential product is a question going back to Gudder’s open-problems list for sequential effect algebras [Gud05]. Gudder and Latréolière [GL08] proved that five conditions, one of which is a trace duality, characterize the Lüders product $a \bullet b = \sqrt{a} b \sqrt{a}$ on Hilbert-space effects. Without the trace duality, the other four conditions no longer determine the product. Liu and Wu [LW09, Rem. 1] exhibited the one-parameter twisted family

$$a \circ_t b = a^{1/2+it} b a^{1/2-it}, \quad t \in \mathbb{R}. \quad (1)$$

Every member of this family satisfies all seven standard axioms and agrees with the Lüders product whenever the first argument is a projection. The algebraic clauses are checked in Liu and Wu’s own proof and in Shen–Wu [SW09, Thm. 3.1, Ex. 3.1]; norm-continuity (S2) follows from continuity of the functional calculus.

Van de Wetering [vdW19] showed every finite-dimensional sequential-product space is order-isomorphic to a Euclidean Jordan algebra; Euclidean Jordan algebras are also the arena of the operational reconstruction program for quantum theory [BGW20]. The question is then how much freedom the axioms leave on a simple one.

The three closest results stop short of answering it. Liu–Wu [LW09] establish the *existence* of the constant-twist family (1). Shen–Wu [SW09] classify the products of a functional-calculus ansatz, under axioms not including (S2) and permitting arbitrary frame dependence at rank two. Van de Wetering [vdW18], on the complex Hilbert-space type and for simple invertible first arguments, reduces the residual comparison automorphism to conjugation by a bicommutant unitary u_a . His own uniqueness theorems reach every type, but each needs an added selector on the product: invariance, inner-product symmetry, or invertibility preservation. None of the three results classifies the admissible dependence $a \mapsto u_a$ under the bare axioms, without an ansatz or an added hypothesis on the product, and none settles the residual freedom across the four simple Euclidean Jordan types of rank at least three (Section 8). Theorem 1 supplies that classification, and Corollary 6.8 identifies the sharp rank-two failure in the complex series as $C(\mathbb{RP}^2, \mathbb{R})$. The freedom the seven axioms leave in a finite-dimensional effect update is therefore at most an outcome-conditioned unitary back-action on each simple type of rank at least three. Where that back-action survives, it shifts sequential measurement probabilities in an explicit single-qubit interferometer (Remark 6.10).

Theorem 1 (Sequential-product classification, rank ≥ 3). *Let J be a finite-dimensional simple Euclidean Jordan algebra of rank $n \geq 3$, and let $(\cdot)$ be any sequential product on its effects—any binary operation $[0, 1]_J \times [0, 1]_J \rightarrow [0, 1]_J$ satisfying the seven axioms (S1)–(S7) of Definition 2.1, including first-variable norm-continuity (S2). Then:*

- *if J is of real type $M_n(\mathbb{R})^{\text{sa}}$, quaternionic type $M_n(\mathbb{H})^{\text{sa}}$, or exceptional type $H_3(\mathbb{O})$, the product is the Lüders product $a \cdot b = Q_{\sqrt{a}} b$;*
- *if J is of complex type $H_n(\mathbb{C}) = M_n(\mathbb{C})^{\text{sa}}$, there is a unique $t \in \mathbb{R}$ with $a \cdot b = a^{1/2+it} b a^{1/2-it}$ for all effects a, b (with $0^{1/2 \pm it} := 0$).*

The complex-type parameter t is stated for the fixed labelled algebra $M_n(\mathbb{C})^{\text{sa}}$, on which it is unique (Remark 5.11); quotienting by all Jordan automorphisms further identifies t with $-t$ through the transpose. Beyond the standing structural hypotheses (finite dimension, simplicity, rank ≥ 3), the seven axioms are the only constraints imposed on the product: Peirce-block-diagonality of the update $a \mapsto a \cdot (\cdot)$ is derived, not assumed through any pair-locality, decomposition-independence, or functional-calculus template (Section 8).

The threshold is sharp in the complex series: Theorem 6.4 exhibits a continuous, frame-dependent product on $M_2(\mathbb{C})^{\text{sa}}$ satisfying all seven axioms and conjugate to no constant-twist member of (1)—a continuous specialization of Shen–Wu’s construction [SW09], here verified against the full axiom list including (S2). (The real qubit is already rigid by Proposition 5.1, and the complex qubit is classified completely: its norm-continuous (S1)–(S7)

products are parametrized bijectively by $C(\mathbb{RP}^2, \mathbb{R})$, one continuous twist value per unordered frame (Corollary 6.8). The rank-two spin factors $\mathbb{R} \oplus \mathbb{R}^k$ beyond the real and complex qubits, including $H_2(\mathbb{H})$ and $H_2(\mathbb{O})$, remain unsettled. Sharpness of the rank-three threshold is therefore witnessed within the complex series, not by a classification of every rank-two type.) The mechanism at the threshold is frame overlap: globalizing the per-frame parameter compares two overlapping frames on the cross-coherence space of a shared third atom (Lemmas 5.4 and 5.6), and a qubit frame has no third atom to share, so the per-frame parameter never globalizes. The four coordinate types exhaust the simple algebras of rank ≥ 3 , so Theorem 1 is a complete typewise classification there. A general sequential-product space is a direct sum of simple algebras (Theorem 2.2), and every product decomposes over that sum (Proposition 3.8)—distinct summands cannot couple—so the simple rank- ≥ 3 case is the whole content apart from the rank- ≤ 2 summands. The one rank-one simple algebra $J = \mathbb{R}$ is immediate: Lemma 3.1(i) gives a positive linear $L_a : \mathbb{R} \rightarrow \mathbb{R}$ and Lemma 3.7 forces $L_a(1) = a$, so $a \cdot b = ab$ is Lüders. Within finite-dimensional Euclidean Jordan algebras under (S1)–(S7), only one classification problem then remains: the other rank-two spin factors. Assembling the pieces gives the following classification.

Theorem 2 (Finite-dimensional classification with resolved rank-two summands). *Let $J = \bigoplus_{\alpha=1}^m J_\alpha$ be a finite-dimensional Euclidean Jordan algebra, decomposed into simple summands, in which every rank-two summand is isomorphic to the real qubit $M_2(\mathbb{R})^{\text{sa}}$ or the complex qubit $M_2(\mathbb{C})^{\text{sa}}$, and let $(\cdot)$ satisfy (S1)–(S7). Fix an isomorphism of each summand onto its model algebra in the Jordan–von Neumann–Wigner list, and read J_α as that model. Then $(\cdot)$ decomposes over the summands, $a \cdot b = \sum_\alpha a_\alpha \cdot b_\alpha$, and on each summand it is:*

- *the Lüders product, if J_α is $\mathbb{R}$, $M_k(\mathbb{R})^{\text{sa}}$ ($k \geq 2$), $M_k(\mathbb{H})^{\text{sa}}$ ($k \geq 3$), or $H_3(\mathbb{O})$;*
- *$a \cdot b = a^{1/2+it_\alpha} b a^{1/2-it_\alpha}$ for a unique $t_\alpha \in \mathbb{R}$, if $J_\alpha = H_k(\mathbb{C})$ with $k \geq 3$;*
- *the frame-dependent product of Proposition 6.3 for a unique continuous $t_\alpha : \mathbb{RP}^2 \rightarrow \mathbb{R}$, if $J_\alpha = M_2(\mathbb{C})^{\text{sa}}$.*

Conversely every independent choice of these parameters defines such a product, so relative to those identifications there is a bijection of sets

$$\mathcal{M}_{\text{SP}}(J) \cong \prod_{\alpha : J_\alpha \text{ complex of rank } \geq 3} \mathbb{R} \times \prod_{\alpha : J_\alpha = M_2(\mathbb{C})^{\text{sa}}} C(\mathbb{RP}^2, \mathbb{R}),$$

every other summand contributing a singleton. Here $\mathcal{M}_{\text{SP}}(J)$ denotes the set of norm-continuous (S1)–(S7) sequential products on the fixed (labelled) effect algebra of J (Remark 5.11).

Proof. The decomposition, and the componentwise assembly giving the converse, are the two halves of Proposition 3.8; the rank- ≥ 3 summands are Theorem 1; the rank-one summand $\mathbb{R}$ is the computation just given; the real qubit is Proposition 5.1 at $n = 2$; the complex qubit is Corollary 6.8. Uniqueness of each parameter on its labelled summand is Remark 5.11 in the first case and the injectivity half of Corollary 6.8 in the second. The converse also needs each listed value to be attained, so that the displayed sets are exact rather than upper bounds. The Lüders product is a norm-continuous (S1)–(S7) product on every summand (Lemma 3.3(i), with (S2) from continuity of the functional calculus), so the singleton factors are nonempty. Every $t_\alpha \in \mathbb{R}$ is attained by Lemma 5.10, and every continuous t_α on $\mathbb{RP}^2$ by Proposition 6.3. $\square$

The complex dial as unitary back-action. The parameter t has an operational reading. Write $s(a)$ for the support projection and $U_a^t := (\mathbb{1} - s(a)) + \sum_{\lambda > 0} \lambda^{it} p_\lambda$ for the unitary from the *support* logarithm of a , so that $U_a^t = a^{it}$ when a is invertible. Then $a^{1/2+it} = \sqrt{a} U_a^t$ and

$$a \circ_t b = \sqrt{a} (U_a^t b (U_a^t)^*) \sqrt{a} = \text{Ad}_{U_a^t}(a \bullet b) : \quad (2)$$

on effects (Heisenberg picture, $\text{Ad}_U(x) := UxU^*$) the complex-type product is a Lüders update followed by conjugation by the outcome-conditioned unitary U_a^t . Dually, the unnormalized state operation is $\rho \mapsto (U_a^t)^*(\sqrt{a} \rho \sqrt{a}) U_a^t$: the Lüders state followed by the opposite unitary a^{-it} . The resulting operator has trace $\text{tr}(a\rho)$, and conditioned on the outcome the normalized posterior divides by $\text{tr}(a\rho)$. Remark 6.10 gives the interferometric signature and the invisibility to sharp first effects. Throughout, the object classified is this effect-level product $(a, b) \mapsto a \cdot b$, not a quantum instrument; the sequential-product/instrument boundary is discussed in Section 8.

Proof inputs and provenance. Van de Wetering’s uniqueness section [vdW18, §5] supplies, for each invertible effect a , the comparison map $\Theta_a := Q_{\sqrt{a}}^{-1} L_a$, a unital linear order isomorphism fixing the *effects* that operator-commute with a ; extending that fixing to the full operator commutant is a span argument given here (Section 3). Two ingredients of the proof are, to our knowledge, not in the prior literature. The first is the verification that this comparison machinery applies without a normality hypothesis in the finite-dimensional EJA setting (through finite spectrality, Lemma 3.3). The second is the frame-stabilizer argument (Sections 4 and 5), which eliminates the parameter on the real, quaternionic, and exceptional types and leaves one scalar direction on the complex torus. The upgrade of Θ_a to a Jordan automorphism is itself imported (the van Imhoff–Roelands order-isomorphism theorem [vIR20, Cor. 2.5]); invoking it in place of the Hilbert-space bicommutant-unitary reduction allows the argument to reach the exceptional type. Section 8 positions the result against prior art, in particular against van de Wetering’s Hilbert-space reduction of the comparison automorphism to a bicommutant unitary. For the complex type in finite dimension and rank ≥ 3 , Theorem 1 pins that residual dependence $a \mapsto \text{Ad}_{u_a}$ to $\text{Ad}_{a^{it}}$ for one global t . The Supplementary Material collects the full input ledger.

2 Preliminaries

2.1 Euclidean Jordan algebras and the Peirce decomposition

A *Euclidean Jordan algebra* (EJA) is a finite-dimensional real commutative (not necessarily associative) algebra $(J, \circ)$ with unit e , equipped with an associative inner product $\langle x, y \rangle = \text{tr}(x \circ y)$, satisfying the Jordan identity $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$. Here tr is the Jordan trace, the sum of eigenvalues in any spectral resolution; on the matrix types it is the ordinary, unnormalized matrix trace. The interior $\Omega = \text{int}(J^+)$ of the cone J^+ of squares is a symmetric cone. We follow Faraut and Korányi [FK94]. The *effect space* is $[0, 1]_J = \{a : 0 \leq a \leq e\}$. For $x \in J$ the multiplication operator is $T_x(y) = x \circ y$ and the quadratic representation is $Q_x = 2T_x^2 - T_{x^2}$; for invertible $a \in \Omega$, $Q_{\sqrt{a}}$ is a cone automorphism [FK94, Prop. III.2.2]. A *Jordan frame* is a complete system of orthogonal primitive idempotents $F = \{p_1, \dots, p_n\}$,

$\sum_i p_i = e$; n is the rank of J . Two elements *operator commute* if $[T_x, T_y] = 0$, equivalently if they are simultaneously diagonalizable in a common Jordan frame.

The Peirce decomposition relative to F is

$$J = \bigoplus_i \mathbb{R}p_i \oplus \bigoplus_{i < j} V_{ij}, \quad V_{ij} = \{x : p_i \circ x = \tfrac{1}{2}x = p_j \circ x\}, \quad (3)$$

with $\dim V_{ij} = d$ independent of $i < j$ when J is simple, the Peirce multiplicity, equal to 1, 2, 4, 8 for $M_n(\mathbb{R})^{\text{sa}}$, $H_n(\mathbb{C})$, $M_n(\mathbb{H})^{\text{sa}}$, $H_3(\mathbb{O})$ respectively [FK94]. (We write $H_n(\mathbb{K})$ and $M_n(\mathbb{K})^{\text{sa}}$ interchangeably for the Hermitian $n \times n$ matrices over $\mathbb{K}$ with the symmetrized product.) For a single idempotent q , $J = J_2(q) \oplus J_1(q) \oplus J_0(q)$ with $J_k(q) = \{x : q \circ x = \frac{k}{2}x\}$; here $J_2(q)$ and $J_0(q)$ are EJAs with units q and $e - q$, and their elements operator commute across the pair. All Jordan frames of a simple EJA are conjugate under $\text{Aut}(J)$ [FK94, Thm. IV.2.5]. By the Jordan spectral theorem every $a \in J$ has a spectral resolution $a = \sum_i \lambda_i p_i$ into a Jordan frame; every effect is therefore *simple* (a finite real combination of orthogonal idempotents; the term applies to elements, not to J).

The standard (Lüders) product on J is $a \bullet b = Q_{\sqrt{a}} b$; on $M_n(\mathbb{C})^{\text{sa}}$ it is $\sqrt{a} b \sqrt{a}$. Three conventions stand throughout. First, e , $\mathbb{1}$ and I all denote the unit. Second, for an effect with spectral resolution $a = \sum_k \lambda_k p_k$, the *support projection* is $[a] := \sum_{\lambda_k > 0} p_k$ (also written $s(a)$), equal to e exactly when a is invertible. Third, every functional-calculus expression takes the value 0 at eigenvalue 0, so in particular $0^{1/2 \pm it} := 0$.

2.2 Sequential products

Following Gudder–Greechie [GG02] and van de Wetering [vdW19]:

Definition 2.1 (Sequential product space, [vdW19, Def. 2]). Let $(V, \leq, \mathbb{1})$ be an order unit space with a binary operation $(\cdot) : [0, 1]_V \times [0, 1]_V \rightarrow [0, 1]_V$. Write $a \mid b$ when $a \cdot b = b \cdot a$. Then $(V, \cdot)$ is a *sequential product space* if:

- (S1) **Additivity:** $a \cdot (b + c) = a \cdot b + a \cdot c$ whenever $b + c \leq \mathbb{1}$.
- (S2) **Continuity:** $a \mapsto a \cdot b$ is continuous in the order unit norm.
- (S3) **Unitality:** $\mathbb{1} \cdot a = a$ for all a .
- (S4) **Symmetry of orthogonality:** $a \cdot b = 0 \Rightarrow b \cdot a = 0$.
- (S5) **Associativity of compatible effects:** $a \mid b \Rightarrow a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
- (S6) **Additivity of compatibility:** $a \mid b \Rightarrow a \mid (\mathbb{1} - b)$; if also $a \mid c$ then $a \mid (b + c)$ when $b + c \leq \mathbb{1}$.
- (S7) **Multiplicativity of compatibility:** $a \mid b$ and $a \mid c \Rightarrow a \mid b \cdot c$.

These are van de Wetering’s Definition 2 [vdW19] restated, under lightly reworded clause names, with the domain riders made explicit; (S2) is his norm-continuity in the first argument, which he adds to the sequential-effect-algebra list and flags as new. His structure theorem places the classification on Euclidean Jordan algebras.

Theorem 2.2 ([vdW19, Thm. 1]). *A finite-dimensional sequential product space is order-isomorphic to a Euclidean Jordan algebra.*

Thus any finite-dimensional order unit space carrying an operation satisfying (S1)–(S7) is (order-isomorphic to) a direct sum of simple Euclidean Jordan algebras,¹ over which the product decomposes with no coupling between summands (Proposition 3.8; the scope consequences are drawn in Section 1). The Jordan–von Neumann–Wigner classification [JvNW34] lists the simple types; those of rank ≥ 3 are $M_n(\mathbb{R})^{\text{sa}}$, $H_n(\mathbb{C})$, $M_n(\mathbb{H})^{\text{sa}}$, and $H_3(\mathbb{O})$.

Lemma 2.3 (Effects span; linear extension). *In an order unit space, $[0, 1]_V$ contains the closed ball of radius $\frac{1}{2}$ about $\frac{1}{2}\mathbb{1}$, hence spans V ; two linear maps agreeing on $[0, 1]_V$ are equal. The same holds for the effect interval $[0, q]$ of the Peirce subalgebra $J_2(q)$, which spans $J_2(q)$.*

Proof. If $\|v\| \leq \frac{1}{2}$ then $0 \leq \frac{1}{2}\mathbb{1} + v \leq \mathbb{1}$; agreement on this ball forces agreement on V by linearity and scaling. Applied to the order unit space $(J_2(q), q)$ this gives the last claim. $\square$

3 The Comparison Map

Fix a finite-dimensional Euclidean Jordan algebra J . Let $(\cdot)$ be a norm-continuous sequential product on $[0, 1]_J$, with left action $L_a(b) = a \cdot b$, and let $a \bullet b = Q_{\sqrt{a}}b$ be the standard reference product, with left action $L_a^0 = Q_{\sqrt{a}}$.

3.1 Simplicity and order preservation

Lemma 3.1 (Positive linear extension and first-variable homogeneity). *For each effect a :* (i) $L_a : b \mapsto a \cdot b$ *is additive by (S1) and order bounded, hence extends uniquely to a positive linear map $L_a : J \rightarrow J$ (Lemma 2.3);* (ii) $(\lambda a) \cdot b = \lambda(a \cdot b)$ *for all $\lambda \in [0, 1]$ and effects b .*

Proof. (i) L_a is additive by (S1) and maps $[0, 1]$ into $[0, 1]$, so it is order bounded. For $x \geq 0$ with $x/n \leq \mathbb{1}$ set $\tilde{L}_a(x) := n L_a(x/n)$; additivity makes this independent of n and additive on J^+ , order boundedness gives real homogeneity, and extension by differences yields a unique positive linear map $L_a : J \rightarrow J$ on $J = \text{span}[0, 1]_J$ (Lemma 2.3). (ii) First-variable homogeneity is van de Wetering’s linearity property (his Proposition 3.9 [vdW18]). His proof passes from rational to real scalars through an infimum. That passage is available because his σ -SEA definition provides both the infimum of a descending sequence and two conditions tying it to the product, $b \cdot \bigwedge a_i = \bigwedge(b \cdot a_i)$ and $b \mid a_i \ \forall i \Rightarrow b \mid \bigwedge a_i$; his proof invokes the second. Here (S2) supplies that passage in its place, as he notes in [vdW18, Prop. 3.9, footnote 2]. The same statement is proved directly from (S1)–(S7), with (S2) in exactly that role, in [vdW19, Prop. 2(3)]. $\square$

Lemma 3.2 (Positive-cone extension). *Each product extends from effect first arguments to arbitrary first arguments v in the positive cone J^+ by*

$$v \cdot b := \mu \left((v/\mu) \cdot b \right), \quad \mu \geq \|v\|, \ \mu > 0,$$

¹The identification of Theorem 2.2 is unital—van de Wetering’s Jordan product takes the order unit as its Jordan unit—so the unit-relative axioms (S1)–(S7) and the effect interval transport to each summand, as Proposition 3.8 requires.

for effects b : the value is independent of the chosen normalization μ , the extension is positively homogeneous in the first argument, and it agrees with the original operation on $[0, 1]_J$.

Proof. If $0 < \mu \leq \mu'$ with $\mu \geq \|v\|$, then v/μ and v/μ' are effects ($0 \leq v$ and $\|v\| \leq \mu$ give $v \leq \mu e$ in an order unit space) and $v/\mu' = (\mu/\mu')(v/\mu)$ with $\mu/\mu' \in (0, 1]$, so Lemma 3.1(ii) gives $\mu'((v/\mu') \cdot b) = \mu'(\mu/\mu')((v/\mu) \cdot b) = \mu((v/\mu) \cdot b)$: any two admissible normalizations agree. Taking $\mu = 1$ for a nonzero $v \in [0, 1]_J$ recovers the original operation (and $0 \cdot b = 0$ by (S1) and (S4), as in Lemma 3.7, consistently with every μ); positive homogeneity $(\lambda v) \cdot b = \lambda(v \cdot b)$ for $\lambda > 0$ follows by normalizing λv with $\lambda\mu$. For the standard product the extension is $Q_{\sqrt{v}}$ itself, by the homogeneity $Q_{\sqrt{\lambda v}} = \lambda Q_{\sqrt{v}}$. $\square$

Lemma 3.3 (Finite-spectral bridge). *Throughout, a simple effect means an effect in van de Wetering's class E_0 [vdW18]: one with a finite spectral decomposition $a = \sum_k \lambda_k p_k$ into orthogonal sharp effects (not to be confused with a simple EJA). Sharpness is order-theoretic— p is sharp when $p \wedge (e - p) = 0$ [vdW18, Def. 3.14]—and under the standard product it is equivalent to $p \bullet p = p$ [vdW18, Prop. 3.15], i.e. $p^2 = p$: the sharp effects of an EJA are exactly its idempotents. On a finite-dimensional EJA J :*

- (i) *the standard product makes $[0, 1]_J$ a convex σ -sequential effect algebra, hence a sequential effect space (SES) [vdW18, Thm. A.6];*
- (ii) *every effect is simple ($E = E_0$), so the simplicity hypothesis of Propositions 5.2, 5.5, 5.7 and 4.20 of [vdW18] is met by every effect, for the unknown product $(\cdot)$ as well (their remaining hypotheses are verified at each use); normality of $(\cdot)$ is never assumed;*
- (iii) *every invertible effect is order preserving for both products (Proposition 4.20 of [vdW18] gives every simple effect a pseudo-inverse, and Proposition 4.19 there that a pseudo-inverse renders the effect order preserving)²;*
- (iv) *the standard product is invariant: for every unital order isomorphism Φ of J , $\Phi(a \bullet b) = \Phi a \bullet \Phi b$ (it is defined from the Jordan product, and unital order isomorphisms of an EJA are Jordan isomorphisms).*

Proof. (i) is van de Wetering's Theorem A.6 [vdW18], stated there for an arbitrary Euclidean Jordan algebra. (ii) holds by the Jordan spectral theorem: every element is a finite real combination of orthogonal idempotents. Consequently the simple-effect clauses cover every effect, and the normality hypothesis needed only to pass to non-simple effects is vacuous. (iii): the pseudo-inverse of a nonzero simple effect has norm ≥ 1 and, unless the effect is sharp, lies outside the effect interval. The positive-cone extension of Lemma 3.2 applies both products to it in the first argument, and the linear extension of Lemma 3.1(i) in the second, independently of the normalization. Proposition 4.20 of [vdW18] supplies a pseudo-inverse for every simple effect from its finite spectral representation $a = \sum_k \lambda_k p_k$ inside the commutative subalgebra of orthogonal sharp effects: explicitly the spectral inverse on the support, $a^- = \sum_{\lambda_k > 0} \lambda_k^{-1} p_k$, with $a \cdot a^- = a^- \cdot a = [a]$ for either product. Proposition 4.19 there turns that pseudo-inverse into order preservation, again for either product. Neither

²A *pseudo-inverse* of a simple effect is its spectral inverse on its support. The *order preserving* property (Definition 4.17 of [vdW18]) has two clauses: the induced left multiplication L_a carries the effect interval onto the order interval below a (lower-interval surjectivity), and it reflects the order relative to the support of a ; for a full-support invertible effect the second clause is order reflection, so L_a is an order embedding onto its image. See [vdW18, Def. 4.17, Props. 4.19–4.20].

proposition is the inverse-law hypothesis of his Theorem 5.19. (iv) is van de Wetering's remark that the Jordan-defined standard product is invariant [vdW18]. $\square$

Lemma 3.4 (Automatic normality). *On a finite-dimensional order-unit space V , any operation $(\cdot)$ on effects satisfying (S1) and (S2) is normal in van de Wetering's sense [vdW18, Defs. 2.3 and 5.1]: for every descending sequence of effects $b_k \downarrow b$ with order-infimum $b = \bigwedge_k b_k$ and every effect a , (i) $a \cdot b_k \downarrow a \cdot b$, and (ii) if $a \mid b_k$ for all k then $a \mid b$.*

Proof. By Lemma 3.1(i)—whose proof uses only the order-unit structure and (S1), and so applies verbatim with V in place of J —the map $L_a : x \mapsto a \cdot x$ is positive linear on V , hence norm-continuous in finite dimension. A descending sequence of effects b_k lies in the compact interval $[0, b_1]$, so it has convergent subsequences. By closedness of the cone every subsequential limit is a lower bound for $\{b_k\}$ and dominates every other lower bound, hence equals the order-infimum $\bigwedge_k b_k = b$. The whole sequence therefore converges: $b_k \rightarrow b$ in norm. (i) Positivity of L_a makes $a \cdot b_k$ descending, and continuity gives $a \cdot b_k \rightarrow a \cdot b$ in norm, so $\bigwedge_k a \cdot b_k = a \cdot b$. (ii) If $a \cdot b_k = b_k \cdot a$ for all k , continuity of L_a sends the left side to $a \cdot b$ and (S2) (first-variable continuity, $b_k \rightarrow b$) the right side to $b \cdot a$, so $a \mid b$. The argument uses only the finite-dimensional order structure with (S1) and (S2), so it is prior to and independent of the Jordan identification. $\square$

Remark 3.5 (The role of (S2)). In finite dimension (S1) and (S2) already make the unknown product *normal* (Lemma 3.4)—derived, never assumed—while Lemma 3.3(ii) independently covers every effect by the simple-effect clauses with no normality extension invoked; Remark 3.11 states both routes and discharges every imported hypothesis on the second, self-contained one. Either way the sole normality-type input is van de Wetering's scalar-homogeneity step (Lemma 3.1(ii)), where (S2) supplies it.

3.2 The compatibility bridge

Proposition 3.6 (Compatibility bridge; [vdW18, Props. A.1 and A.3]). *For effects a, b of a Euclidean Jordan algebra, with the standard product $a \bullet b = Q_{\sqrt{a}}b$,*

$$a \bullet b = b \bullet a \iff [Q_a, Q_b] = 0 \iff [T_a, T_b] = 0,$$

i.e. standard-product compatibility is exactly Jordan operator commutation. In particular simultaneously diagonalizable effects are compatible for the standard product.

The two equivalences are van de Wetering's Propositions A.1 and A.3 [vdW18]; the second ($[Q_a, Q_b] = 0 \iff [T_a, T_b] = 0$) is the one he notes he could not find in the literature and which is treated in [vdW20]. Combined with Proposition 5.2 of [vdW18] (for simple a, b , standard-product compatibility transfers the product value and the compatibility to the unknown product), simultaneously diagonalizable effects are compatible for the unknown product and their two products agree.

Lemma 3.7 (The unit as second argument). *$a \cdot \mathbb{1} = a$ for every effect a .*

Proof. Using (S1), (S3), (S4), (S6): by (S1) with $b = c = 0$, $a \cdot 0 = 0$; by (S4), $0 \cdot a = 0$; so $a \mid 0$; by (S6), $a \mid 0^\perp = \mathbb{1}$; and by (S3) with compatibility, $a \cdot \mathbb{1} = \mathbb{1} \cdot a = a$. $\square$

The reduction to a simple algebra is itself a consequence of the axioms.

Proposition 3.8 (Central decomposition). *Let $J = \bigoplus_{\alpha=1}^m J_\alpha$ be any finite-dimensional Euclidean Jordan algebra with central summand units e_α , and let $(\cdot)$ be a sequential product satisfying (S1)–(S7). For an effect a write $a_\alpha := e_\alpha \circ a \in [0, 1]_{J_\alpha}$. Then*

$$a \cdot b = \sum_{\alpha=1}^m a_\alpha \cdot b_\alpha, \quad a_\alpha \cdot b_\alpha \in J_\alpha,$$

and each restriction is a sequential product satisfying (S1)–(S7) on the order-unit space $(J_\alpha, \leq, e_\alpha)$. A general finite-dimensional sequential product is thus the direct sum of its summand products; distinct simple summands cannot couple. Conversely, given any sequential products $(\cdot)_\alpha$ satisfying (S1)–(S7) on the summands, $a \cdot b := \sum_\alpha (a_\alpha \cdot_\alpha b_\alpha)$ satisfies (S1)–(S7) on J and restricts to them.

Proof. The bridge (Proposition 3.6) and the Proposition 5.2 transfer of value and compatibility hold on any finite-dimensional EJA, using only that the effects involved are simple (Jordan spectral theorem), not that J is simple; the comparison map Θ_a is not used. Each e_α is central, hence operator-commutes with every effect, so a and e_α are compatible for the unknown product and $a \cdot e_\alpha = a \bullet e_\alpha = Q_{\sqrt{a}} e_\alpha = a_\alpha$; likewise $e_\alpha \cdot b_\alpha = Q_{\sqrt{e_\alpha}} b_\alpha = b_\alpha$ for $b_\alpha \leq e_\alpha$. By (S5) with $a \mid e_\alpha$,

$$a \cdot b_\alpha = a \cdot (e_\alpha \cdot b_\alpha) = (a \cdot e_\alpha) \cdot b_\alpha = a_\alpha \cdot b_\alpha,$$

and monotonicity of the positive map L_a (Lemma 3.1) gives $0 \leq a \cdot b_\alpha \leq a \cdot e_\alpha = a_\alpha \leq e_\alpha$, so it lies in $[0, e_\alpha] \subseteq J_\alpha$. Summing over α by (S1) gives the formula. Block-diagonality makes compatibility componentwise: for $x, y \in [0, 1]_{J_\alpha}$ their compatibility in J_α coincides with their global compatibility, and the component formula carries the global complement $\mathbb{1} - y$ to the summand complement $e_\alpha - y$ inside J_α . Explicitly, $\mathbb{1} - y = (e_\alpha - y) + \sum_{\beta \neq \alpha} e_\beta$, and since $x \in [0, 1]_{J_\alpha}$ the component formula gives $x \cdot (\mathbb{1} - y) = x \cdot (e_\alpha - y)$ and $(\mathbb{1} - y) \cdot x = (e_\alpha - y) \cdot x$, so global compatibility of x with $\mathbb{1} - y$ is compatibility with the summand complement $e_\alpha - y$. Hence the complement clause of (S6) restricts to each summand; its additivity clause and (S1)–(S5), (S7) restrict directly, all on $(J_\alpha, \leq, e_\alpha)$, whose order-unit norm is the restriction of the direct-sum norm. For the converse, every axiom is verified coordinate by coordinate. Effects, the order, and complements are componentwise, giving (S1), (S3), (S4). Compatibility $a \mid b$ holds in J exactly when $a_\alpha \mid b_\alpha$ for every α , so (S5)–(S7) reduce to the summand clauses. Finally (S2) holds because the direct-sum order-unit norm is $\max_\alpha \|\cdot\|_\alpha$, so norm convergence is convergence in every coordinate. $\square$

The classification over J is therefore exactly the product over α of the summand classifications.

3.3 The comparison map is a Jordan automorphism

The comparison-map construction needs one hypothesis at every invertible effect: order preservation *for the unknown product as well as the standard one*. Lemma 3.3(iii) records this; because every argument below depends on that hypothesis, we display the transfer step rather than leave it to be reassembled from the imported propositions.

Proposition 3.9 (Pseudo-inverses transfer to the unknown product). *Let a be an invertible effect of a finite-dimensional EJA J , with spectral inverse $a^{-1} \in J^+$. Then*

$$a \cdot a^{-1} = a^{-1} \cdot a = \mathbb{1}$$

for the unknown product as well as the standard one, so a^{-1} is a pseudo-inverse of a for both; consequently a is order preserving for both, and the hypotheses of Proposition 5.3 of [vdW18] hold at every invertible effect.

Proof. Fix $\mu \geq \|a^{-1}\|$, so that a^{-1}/μ is an effect. Being a polynomial in a , it is diagonal in a spectral frame of a , hence operator commutes with a and so is compatible with it for the standard product (Proposition 3.6). Both are effects of a finite-dimensional EJA, hence simple (Lemma 3.3(ii)), so Proposition 5.2 of [vdW18] applies to the pair in either order: they are compatible for the unknown product too, and there the product takes its standard value. Undo the normalization on each side. In the second argument L_a is positive linear (Lemma 3.1(i)), so

$$a \cdot a^{-1} = \mu a \cdot (a^{-1}/\mu) = \mu a \bullet (a^{-1}/\mu) = a \bullet a^{-1} = Q_{\sqrt{a}} a^{-1} = \mathbb{1}.$$

In the first argument the positive-cone extension is positively homogeneous and independent of the normalization (Lemma 3.2), so likewise $a^{-1} \cdot a = \mu (a^{-1}/\mu) \cdot a = \mu (a^{-1}/\mu) \bullet a = a^{-1} \bullet a = Q_{\sqrt{a^{-1}}} a = \mathbb{1}$. Thus a^{-1} is a pseudo-inverse of a for both products, and Proposition 4.19 of [vdW18] makes a order preserving for both. Note the order of dependence: the transfer uses only compatibility for the *standard* product together with simplicity of the two effects, so nothing here presupposes the comparison map or any conclusion drawn from it. $\square$

Proposition 3.10 (Comparison map; [vdW18, §5] + [vIR20, Cor. 2.5]). *For every invertible effect a of J there is a Jordan automorphism Θ_a of J with*

$$a \cdot b = a \bullet \Theta_a(b) = Q_{\sqrt{a}} \Theta_a(b) \quad \text{for all } b,$$

equivalently $L_a = Q_{\sqrt{a}} \Theta_a$. Moreover Θ_a fixes every b that operator-commutes with a ; and for operator-commuting invertible a, a' , $\Theta_{a \bullet a'} = \Theta_a \Theta_{a'} = \Theta_{a'} \Theta_a$.

Proof. Proposition 5.3 of [vdW18] constructs a *unital linear order isomorphism* Θ_a of J with $L_a = Q_{\sqrt{a}} \Theta_a$ (its multiplication-operator form; $Q_{\sqrt{a}}$ is invertible since a is invertible). Linearity and the order-iso property come with Proposition 5.3's construction (its Θ_a arises by the universal property of the quotient of his Proposition 4.22, whose maps are positive and linear); unitality is $\Theta_a(\mathbb{1}) = Q_{\sqrt{a}}^{-1}(a \cdot \mathbb{1}) = Q_{\sqrt{a}}^{-1}(a) = \mathbb{1}$, using $a \cdot \mathbb{1} = a$ (Lemma 3.7) and $Q_{\sqrt{a}}(\mathbb{1}) = a$. Since J is a finite-dimensional JB-algebra, the unital linear order isomorphism Θ_a is a Jordan automorphism (van Imhoff–Roelands [vIR20, Cor. 2.5]; classically Alfsen–Shultz [AS03], cited for exactly this by van de Wetering [vdW18]). Thus $\Theta_a \in \text{Aut}(J)$.

The fixing property is Proposition 5.5 of [vdW18]: for b compatible with a under the standard product (which by Proposition 3.6 is operator commutation) and $b \leq [a] = e$ (a being invertible), $\Theta_a(b) = b$. Proposition 5.5 needs compatibility under the *reference* product only; that a and b are then also compatible for the unknown product is a conclusion (Proposition 5.2 of [vdW18]), not a second hypothesis. This fixing extends from the effects

$b \leq e$ to the whole operator commutant of a by linearity: the commutant is an order-unit subspace, linearly spanned by its unit-interval effects, on each of which Θ_a is the identity. The cocycle law is Proposition 5.7 of [vdW18], whose hypothesis is invariance of the *reference* product (Lemma 3.3(iv)); we impose no invariance on the unknown product. $\square$

Proposition 3.10 reduces the classification to a single Jordan automorphism: every type-dependent step below turns on a property of Θ_a . The proposition’s inputs are all imported; the provenance split is stated in Section 1 and the imported hypotheses are discharged in Remark 3.11.

Remark 3.11 (Hypotheses of the imported propositions, discharged in place). The comparison propositions invoked above (his 5.2, 5.3, 5.5, 5.7) have their simplicity and normality hypotheses discharged here on two independent grounds: the unknown product is *normal* (Lemma 3.4), as is the standard one (Lemma 3.3(i)), so van de Wetering’s normality-based conclusions—which ask normality of both products—apply to every effect directly; and, independently, every effect of a finite-dimensional EJA is simple ($E = E_0$, Lemma 3.3(ii)), so the normality-free simple-effect clauses apply to every effect as well—not through simplicity of the ambient EJA. (Of the remaining rows, those for Propositions 4.19–4.20 and Corollary 2.5 are normality-free; the row for his Proposition 3.9 carries the sole normality-type input, supplied in its place by (S2)—see Remark 3.5.) The table discharges every hypothesis on the second, self-contained route. We write $(\cdot)$ for the unknown product and $a \bullet b = Q_{\sqrt{a}} b$ for the standard reference product; the comparison automorphism acts on the reference argument, $L_a = Q_{\sqrt{a}} \Theta_a$, as in [vdW18].

Imported result	Hypotheses	Discharged by
Prop. 3.9 of [vdW18]: first-variable scalar homogeneity	a sequential effect space; σ -normality, or first-variable norm-continuity in its place	(S2) (Remark 3.5); the same statement is proved from (S1)–(S7) directly in [vdW19, Prop. 2(3)]
Props. 4.19–4.20 [vdW18]: simple effects have pseudo-inverses, and a pseudo-inverse makes the effect order preserving	a a simple effect	$E = E_0$ (Lemma 3.3(ii)); pseudo-inverse reached by Lemma 3.2
Prop. 5.2 [vdW18]: transfer of value and compatibility	a, b simple effects in a sequential effect space, compatible for the <i>reference</i> product	Lemma 3.3(i)–(ii); reference compatibility = operator commutation (Proposition 3.6), checked at each use
Prop. 5.3 [vdW18]: the comparison map Θ_a	a order preserving <i>for both products</i>	Lemma 3.3(iii)
Prop. 5.5 [vdW18]: commutant fixing	a, b simple; $b \leq [a]$; reference-product compatibility	$E = E_0$; a invertible, so $[a] = e$; Proposition 3.6
Prop. 5.7 [vdW18]: cocycle law	commuting invertible simple effects; <i>reference</i> product invariant (his Def. 5.6)	$E = E_0$; Lemma 3.3(iv); operator commutation verified at each use
Cor. 2.5 [vIR20]: unital linear order isomorphism $\Rightarrow$ Jordan automorphism	a unital linear bijection of a JB-algebra	J a finite-dimensional EJA, hence a JB-algebra

Two standing clarifications. None of the cited propositions assumes the *ambient* algebra is simple—only that the effects involved are; this licenses Proposition 3.8 inside a non-simple direct sum. And Proposition 5.3’s extension to all of J for invertible a uses only that pseudo-inverses are product-independent on simple effects. (Two further clarifications—unknown-product compatibility as Proposition 5.2’s *conclusion*, and no invariance imposed on the unknown product—are recorded in the proof of Proposition 3.10.)

Remark 3.12 (No Hilbert-space unitary is needed). Van de Wetering’s Lemma 5.8 further reduces Θ_a , for simple invertible a , to Ad_{u_a} for a bicommutant unitary u_a , using that all automorphisms of $B(H)$ are inner. That step is Hilbert-space-specific and has no octonionic analogue. (The proof of his Theorem 5.11 states that the Theorem 5.9 argument carries over to $B(H)^{\text{sa}}$ for real, complex, quaternionic or octonionic H . The three associative types are indeed covered; the Albert algebra is not the self-adjoint part of any associative algebra, and $\text{Aut}(H_3(\mathbb{O})) = F_4$ is not a group of inner automorphisms, so the bicommutant unitary is unavailable there. Theorem 5.3 below reaches the exceptional type by a different route.) We do not use the step: the frame-stabilizer argument below needs only that Θ_a is a Jordan automorphism, which Proposition 3.10 supplies on every type.

4 The Common Reduction

Fix a simple EJA J of rank $n \geq 2$, a Jordan frame $F = \{p_1, \dots, p_n\}$, and for $r \in (-\infty, 0]^n$ the invertible effect $a(r) = \sum_i e^{r_i} p_i$ diagonal in F . Write $\Theta_r := \Theta_{a(r)}$. The lemmas of this section hold at every rank; the rank hypothesis $n \geq 3$ enters only in the typewise branches of Section 5.

Lemma 4.1 (Frame fixing; block-diagonality is a conclusion). *Θ_r fixes each p_i and the diagonal $\bigoplus_i \mathbb{R}p_i$ pointwise, preserves each Peirce block V_{ij} (the common $\frac{1}{2}$ -eigenspace of T_{p_i} and T_{p_j}), and lies in the identity component $\text{Stab}(F)^\circ$ of the frame stabilizer $\text{Stab}(F) = \{g \in \text{Aut}(J) : gp_i = p_i\}$. Consequently $L_{a(r)} = Q_{\sqrt{a(r)}} \Theta_r$ is Peirce-block-diagonal.*

Proof. Each p_i is diagonal in F , hence operator-commutes with $a(r)$; by Proposition 3.10 (with Proposition 3.6), $\Theta_r(p_i) = p_i$, and by linearity $\Theta_r = \text{id}$ on $\bigoplus_i \mathbb{R}p_i$. A Jordan automorphism fixing every p_i preserves each V_{ij} : for $x \in V_{ij}$, $p_i \circ \Theta_r(x) = \Theta_r(p_i \circ x) = \frac{1}{2} \Theta_r(x)$ and likewise for p_j , so $\Theta_r(x) \in V_{ij}$. Thus $\Theta_r \in \text{Stab}(F)$. For continuity of $r \mapsto \Theta_r$, fix an effect basis $b_1, \dots, b_m$ of J (Lemma 2.3). (S2) makes each $a \mapsto L_a(b_k)$ norm-continuous, so the matrix coefficients of L_a in this basis are continuous, giving operator-norm continuity of $a \mapsto L_a$ in finite dimension. Since $a \mapsto Q_{\sqrt{a}}^{-1} = Q_{a^{-1/2}}$ is continuous on invertibles as well, $r \mapsto \Theta_r = Q_{a(r)^{-1/2}} L_{a(r)}$ is continuous. With the path $t \mapsto \Theta_{tr}$ from $\Theta_0 = \text{id}$ this places $\Theta_r \in \text{Stab}(F)^\circ$; $\text{Stab}(F) = \bigcap_i \{g \in \text{Aut}(J) : gp_i = p_i\}$ is closed in the compact Lie group $\text{Aut}(J)$, hence a closed Lie subgroup. Block-diagonality of $L_{a(r)}$ follows from $L_{a(r)} = Q_{\sqrt{a(r)}} \Theta_r$, both factors block-diagonal. $\square$

Lemma 4.2 (Coalescence). *If $r_i = r_j$, then Θ_r fixes V_{ij} pointwise. More generally, if an invertible effect a has the form $a = \lambda q + a_0$ with $q = p_i + p_j$, $\lambda > 0$, and $a_0 \in J_0(q)$ invertible in $J_0(q)$ —that is, if a is scalar on the Peirce 2-space of q and has no Peirce 1-part—then $\Theta_a|_{J_2(q)} = \text{id}$.*

Proof. Let $q = p_i + p_j$ and $a = \lambda q + a_0$, $a_0 \in J_0(q)$. Take an effect b with $0 \leq b \leq q$, so $b \in J_2(q)$. Spectrally decompose $b = \sum_{\alpha} \mu_{\alpha} e_{\alpha}$ ($\sum_{\alpha} e_{\alpha} = q$) inside $J_2(q)$ and $a_0 = \sum_{\beta} \nu_{\beta} f_{\beta}$ ($\sum_{\beta} f_{\beta} = e - q$) inside $J_0(q)$. Every e_{α} is orthogonal to every f_{β} , so the combined family is a complete orthogonal system of idempotents (refining to a Jordan frame of J), and $a = \sum_{\alpha} \lambda e_{\alpha} + \sum_{\beta} \nu_{\beta} f_{\beta}$ is diagonal in it; hence a and b are simultaneously diagonalizable and operator commute. By Proposition 3.6 they are compatible for the standard product, and since a is invertible, $[a] = e \geq b$, so Proposition 3.10 gives $\Theta_a(b) = b$. Effects supported by q span $J_2(q)$ (Lemma 2.3), so $\Theta_a|_{J_2(q)} = \text{id}$; in particular Θ_a fixes $V_{ij} \subset J_2(q)$. The first statement is the case $a = a(r)$ with $r_i = r_j$. The argument is spectral Peirce calculus, valid for every coordinate type including the exceptional one. $\square$

Lemma 4.3 (Diagonal homomorphism and hyperplane factorization). *The map $r \mapsto \Theta_r$ is a continuous homomorphism $((-\infty, 0]^n, +) \rightarrow \text{Stab}(F)^{\circ}$; it extends to a continuous homomorphism $\chi : (\mathbb{R}^n, +) \rightarrow \text{Stab}(F)^{\circ}$ with a real-linear differential $d\chi : \mathbb{R}^n \rightarrow \mathfrak{stab}(F)$, and the block representation $\rho_{ij} : \mathfrak{stab}(F) \rightarrow \mathfrak{so}(V_{ij})$ —the differential of the group action $\text{Stab}(F)^{\circ} \rightarrow \text{SO}(V_{ij})$, at which Lie-algebra level all faithfulness claims below are made—satisfies*

$$\rho_{ij}(d\chi(r)) = (r_i - r_j) T_{ij} \quad \text{for a single } T_{ij} \in \mathfrak{so}(V_{ij}).$$

Proof. For operator-commuting diagonal $a(r), a(r')$ the standard product satisfies $a(r) \bullet a(r') = a(r + r')$, so Proposition 5.7 of [vdW18] (Proposition 3.10) gives $\Theta_{r+r'} = \Theta_r \Theta_{r'}$; the image is abelian. Extend to $\mathbb{R}^n = S - S$, $S = (-\infty, 0]^n$, by $\chi(s - t) := \Theta_s \Theta_t^{-1}$, well defined by commutativity ($s - t = s' - t' \Rightarrow s + t' = s' + t \Rightarrow \Theta_s \Theta_{t'} = \Theta_{s'} \Theta_t \Rightarrow \Theta_s \Theta_t^{-1} = \Theta_{s'} \Theta_{t'}^{-1}$, the last step because the image is abelian) and continuous via the canonical continuous decomposition $x = s(x) - t(x)$ with $s_i(x) = \min(x_i, 0)$, $t_i(x) = \min(-x_i, 0)$. A continuous homomorphism between finite-dimensional Lie groups is smooth, so $d\chi : \mathbb{R}^n \rightarrow \mathfrak{stab}(F)$ is real-linear. A continuous homomorphism out of the connected group $\mathbb{R}^n$ is moreover determined by its differential, so recovering $d\chi$ recovers χ and each Θ_r (used in the branches to pass from $d\chi = 0$ to $\Theta_r = \text{id}$). For $h \in S$ with $h_i = h_j$, Lemma 4.2 gives $\chi(sh)|_{V_{ij}} = \text{id}$ for all $s \geq 0$; right-differentiating at $s = 0$ yields $\rho_{ij}(d\chi(h)) = 0$. Differences of such h span the hyperplane $\{r_i = r_j\}$: any v with $v_i = v_j$ is $v = h - h'$ with $h = v - t(1, \dots, 1)$, $h' = -t(1, \dots, 1)$, $t \geq \max(0, \max_k v_k)$, both in the orthant and both satisfying the equality. The real-linear map $r \mapsto \rho_{ij}(d\chi(r))$ therefore vanishes on that hyperplane and is a multiple of $(r_i - r_j)$: $\rho_{ij}(d\chi(r)) = (r_i - r_j) T_{ij}$ with $T_{ij} \in \mathfrak{so}(V_{ij})$ (the stabilizer lies in $\text{Aut}(J)$, hence preserves the trace form and acts orthogonally on each block, as in Proposition 4.4). At the level of the differential, no global logarithm and no 2π ambiguity enter. $\square$

Throughout, $\text{Stab}(F)$ is the *pointwise* frame stabilizer (fixing each p_i), not the larger setwise stabilizer that also permutes the atoms.

Proposition 4.4 (Frame-stabilizer data). *Let J be a simple EJA of rank n of one of the four types listed below—these exhaust the simple EJAs of rank ≥ 3 , though not those of rank two—and $\text{Stab}(F)$ the pointwise stabilizer of a Jordan frame. As $\text{Stab}(F) \subset \text{Aut}(J)$ preserves the trace form $\langle x, y \rangle = \text{tr}(x \circ y)$, it acts on each Peirce block V_{ij} through $O(V_{ij})$, and $\mathfrak{stab}(F)$ through $\mathfrak{so}(V_{ij})$. The identity component $\text{Stab}(F)^{\circ}$ and the induced block action are, by type:*

J	$d = \dim V_{ij}$	$\text{Stab}(F)^\circ$	$\mathfrak{stab}(F)$ on V_{ij}
$M_n(\mathbb{R})^{\text{sa}}$	1	$\{1\}$	0
$H_n(\mathbb{C})$	2	$U(1)^n/U(1) \cong T^{n-1}$	$z \mapsto i(\theta_i - \theta_j)z$
$M_n(\mathbb{H})^{\text{sa}}$	4	$Sp(1)^n/\{\pm 1\}$	$x \mapsto \xi_i x - x \xi_j, \xi_\bullet \in \text{Im } \mathbb{H}$
$H_3(\mathbb{O})$	8	$\text{Spin}(8)$	triality: the $8_v, 8_s, 8_c$ differentials

Proof. Trace-form invariance of $\text{Aut}(J)$ is standard [FK94, Ch. III], giving the orthogonality statement. For the matrix types the identity component of $\text{Aut}(M_n(\mathbb{K})^{\text{sa}})$ acts by $X \mapsto UXU^*$. Fixing every $p_i = E_{ii}$ forces $U = \text{diag}(u_i)$ with $u_i \in \mathbb{K}$, $|u_i| = 1$, so the induced action on $V_{ij} \cong \mathbb{K}$ is $x \mapsto u_i x \bar{u}_j$: real ($u_i = \pm 1$, discrete), complex ($U(1)^n/U(1)$, with differential $z \mapsto i(\theta_i - \theta_j)z$), and quaternionic ($Sp(1)^n/\{\pm 1\}$, differential $x \mapsto \xi_i x - x \xi_j$; the quotient by the discrete central $\{\pm 1\}$ does not change the Lie algebra $(\text{Im } \mathbb{H})^n$). For $H_3(\mathbb{O})$ the pointwise stabilizer is $\text{Spin}(8)$ acting on V_{12}, V_{13}, V_{23} by the three inequivalent triality representations $8_v, 8_s, 8_c$, in some order (Yokota [Yok09, Thms. 2.7.1, 1.16.2]; see also [SV00]). The data are otherwise classical [FK94, Loo77]. $\square$

The real stabilizer is discrete, so its identity component is trivial; for the complex type the transpose lies outside the connected torus and plays no role. The remaining branches turn on different representation-theoretic mechanisms. In the quaternionic type the two-slot map $(\xi_i, \xi_j) \mapsto (x \mapsto \xi_i x - x \xi_j)$ is injective, so it separates the two atoms of a coherence block—even though for $n \geq 3$ an individual ρ_{ij} is *not* faithful there, the spectator coordinates lying in its kernel. In the exceptional type each block differential is a nontrivial representation of the simple Lie algebra $\mathfrak{spin}(8)$, hence faithful, so a single block already detects the Lie-algebra element. Only for $\mathbb{C}$ does the action retain a scalar direction (it sees θ_i, θ_j only through their difference), and there too, for $n \geq 3$, ρ_{ij} is not individually faithful. Hence at rank ≥ 3 only the complex type survives.

5 The Typewise Branches

We run the frame-stabilizer reduction of Section 4 type by type; by Proposition 4.4 the stabilizer block data collapse the residual freedom, at rank ≥ 3 , on every type but the complex, which retains one scalar direction.

Proposition 5.1 (Real type). *On $M_n(\mathbb{R})^{\text{sa}}$, $n \geq 2$: $\Theta_r = \text{id}$ for all r , so $a \cdot b = Q_{\sqrt{a}}b$. In particular the real qubit ($n = 2$) is rigid.*

Proof. The pointwise frame stabilizer $\{\pm 1\}^n/\{\pm 1\}$ is discrete at every rank $n \geq 2$, so $\text{Stab}(F)^\circ = \{1\}$; the continuous homomorphism χ from connected $\mathbb{R}^n$ into it is trivial, giving $\Theta_r = \text{id}$ and $L_{a(r)} = Q_{\sqrt{a(r)}}$ for every invertible $a(r)$. Every invertible effect is diagonal in its own frame, so this holds for all invertible effects, and for all effects by Lemma 5.9. $\square$

Theorem 5.2 (Quaternionic type). *On $M_n(\mathbb{H})^{\text{sa}}$, $n \geq 3$: $\Theta_r = \text{id}$ for all r , so $a \cdot b = Q_{\sqrt{a}}b$.*

Proof. Here $\mathfrak{stab}(F) = (\text{Im } \mathbb{H})^n$ and, by Lemma 4.3, $d\chi(r) = (\xi_1(r), \dots, \xi_n(r))$ with $\xi_i(r) = \sum_l r_l a_{il}$, $a_{il} \in \text{Im } \mathbb{H}$, acting on $V_{ij} \cong \mathbb{H}$ by $x \mapsto \xi_i(r)x - x \xi_j(r)$, pinned to $(r_i - r_j)T_{ij}$:

$$\xi_i(r)x - x \xi_j(r) = (r_i - r_j)T_{ij}(x) \quad \forall x, \forall r. \quad (4)$$

Fix $i \neq l$; since $n \geq 3$ choose $j \notin \{i, l\}$. Matching the coefficient of r_l in (4) for block (i, j) : the right side is $\propto (r_i - r_j)$ with $l \neq i, j$, so its r_l -coefficient is 0, giving $a_{il}x = xa_{jl}$ for all x ; setting $x = 1$, $a_{il} = a_{jl}$, then $a_{il}x = xa_{il}$ forces $a_{il} \in Z(\mathbb{H}) \cap \text{Im } \mathbb{H} = \{0\}$. So $a_{il} = 0$ for $i \neq l$. Matching r_i gives $T_{ij}(x) = a_{ii}x$ and r_j gives $T_{ij}(x) = xa_{jj}$, so $a_{ii}x = xa_{jj}$ for all x , forcing $a_{ii} = a_{jj} \in Z(\mathbb{H}) \cap \text{Im } \mathbb{H} = \{0\}$, whence $T_{ij} = 0$ and $d\chi = 0$. Thus $\Theta_r = \text{id}$ for every frame and $a \cdot b = Q_{\sqrt{a}}b$ on invertibles, hence (Lemma 5.9) everywhere. $\square$

Theorem 5.3 (Exceptional type). *On $H_3(\mathbb{O})$: $\Theta_r = \text{id}$ for all r , so $a \cdot b = Q_{\sqrt{a}}b$.*

Proof. The pointwise frame stabilizer is $\{\alpha \in F_4 = \text{Aut}(H_3(\mathbb{O})) : \alpha E_i = E_i, i = 1, 2, 3\} \cong \text{Spin}(8)$ (Yokota [Yok09, Thm. 2.7.1]; see also [SV00]), realized as the triality triple $\{(a_1, a_2, a_3) \in SO(8)^3 : (a_1x)(a_2y) = a_3(xy)\}$ (Yokota [Yok09, Thm. 1.16.2]). Its three $SO(8)$ factors act on V_{12}, V_{13}, V_{23} as the three inequivalent 8-dimensional representations $8_v, 8_s, 8_c$ of $\mathfrak{spin}(8)$ [Bae02, §2.4, 4.2], in some order; the labelling is permuted by the outer automorphisms, and only nontriviality is used below. By Lemma 4.3, $d\chi(r) = \sum_{k=1}^3 r_k \eta_k$ with $\eta_k \in \mathfrak{spin}(8)$ and $\rho_{ij}(d\chi(r)) = (r_i - r_j)T_{ij}$, the $\rho_{12}, \rho_{13}, \rho_{23}$ being these three triality representations. Each is a nontrivial representation of the simple Lie algebra $\mathfrak{spin}(8)$, hence *faithful*—unlike the individual quaternionic and complex block representations at rank ≥ 3 , and unlike the three *group* representations $8_v, 8_s, 8_c$, each of which has a two-element central kernel; the argument uses only the differential. For block V_{12} , the coefficient of r_3 gives $\rho_{12}(\eta_3) = 0$ (coalescence, $\text{RHS} \propto r_1 - r_2$), so $\eta_3 = 0$; symmetrically $\eta_2 = \eta_1 = 0$. Thus $d\chi \equiv 0$, $\Theta_r = \text{id}$. Rank is exactly 3, so each frame is directly a three-index configuration and no globalization is needed; all frames are conjugate [FK94, Thm. IV.2.5]. Working with $\text{Spin}(8)$ abstractly sidesteps octonionic non-associativity. Hence $a \cdot b = Q_{\sqrt{a}}b$ on invertibles, and (Lemma 5.9) everywhere. $\square$

To globalize the complex per-frame parameter across overlapping frames (Lemma 5.6), we need a generator on the shared cross-coherence space that does not depend on how the shared rank-two block splits into atoms; the next lemma constructs it.

Lemma 5.4 (Splitting-independent complex structure on a cross-coherence space). *Let $J = M_n(\mathbb{C})^{\text{sa}}$, $n \geq 3$, let q be a rank-two projection and p_k an atom with $q \circ p_k = 0$, and let $X = \{x : q \circ x = \frac{1}{2}x = p_k \circ x\}$ be the cross-coherence space, identified with the self-adjoint part of $qM_n(\mathbb{C})p_k \oplus p_kM_n(\mathbb{C})q$: each $x \in X$ is $x = z + z^*$ with corner $z = qxp_k$. Define the real-linear map*

$$\mathcal{J}_{q,k}(x) = iz - i(z^*), \quad z = qxp_k.$$

Then:

- (i) $\mathcal{J}_{q,k}$ maps X to X with $\mathcal{J}_{q,k}^2 = -\mathbf{1}_X$, and depends only on q, p_k , and the fixed ambient complex scalar i of the labelled matrix algebra—not on any splitting $q = p_1 + p_2$ into atoms.
- (ii) Every inner automorphism Ad_u (u unitary) with $uqu^* = q$ and $up_ku^* = p_k$ commutes with $\mathcal{J}_{q,k}$ on X .
- (iii) Let $\{p_l\}_{l \geq 3}$ be pairwise-orthogonal atoms with $q + \sum_{l \geq 3} p_l = e$ and p_k among them, and $a = \lambda q + \sum_{l \geq 3} \lambda_l p_l$ with $\lambda, \lambda_l \in (0, 1)$. Then for every $t \in \mathbb{R}$,

$$\text{Ad}_{a^{it}}|_X = \exp(t(\log \lambda - \log \lambda_k) \mathcal{J}_{q,k}),$$

a rotation of X in the orientation fixed by $\mathcal{J}_{q,k}$.

Proof. The corner $z = qxp_k$ is uniquely determined by x since q, p_k are fixed. As $(iz^*)^* = -iz$, one has $(iz - iz^*)^* = -iz^* + iz$, so $\mathcal{J}_{q,k}(x)$ is self-adjoint and lies in $qM_n(\mathbb{C})p_k \oplus p_kM_n(\mathbb{C})q$, i.e. in X ; its corner is iz . Applying $\mathcal{J}_{q,k}$ again, $\mathcal{J}_{q,k}^2(x) = i(iz) - i(iz)^* = -z - z^* = -x$, giving (i); only q, p_k, i enter the definition. For (ii), $uqu^* = q$ and $up_ku^* = p_k$ give $q(uxu^*)p_k = u(qxp_k)u^* = uz u^*$, so $\mathcal{J}_{q,k}(\text{Ad}_u x) = iuz u^* - i(uz u^*)^* = u(iz - iz^*)u^* = \text{Ad}_u(\mathcal{J}_{q,k}x)$. For (iii), a is invertible, so $a^{it} = e^{it \log \lambda} q + \sum_{l \geq 3} e^{it \log \lambda_l} p_l$ is unitary; since $z = qzp_k$, $\text{Ad}_{a^{it}}(x)$ has corner $a^{it} z (a^{it})^* = e^{it \log \lambda} z e^{-it \log \lambda_k} = e^{i\varphi} z$ with $\varphi = t(\log \lambda - \log \lambda_k)$. On X , multiplying the corner by $e^{i\varphi} = \cos \varphi + i \sin \varphi$ is $\cos \varphi \mathbb{1}_X + \sin \varphi \mathcal{J}_{q,k} = \exp(\varphi \mathcal{J}_{q,k})$, using $\mathcal{J}_{q,k}^2 = -\mathbb{1}_X$. $\square$

Proposition 5.5 (One parameter per frame). *Let $J = H_n(\mathbb{C}) = M_n(\mathbb{C})^{\text{sa}}$, $n \geq 3$, and let $F = \{p_1, \dots, p_n\}$ be a Jordan frame. There is a single $t_F \in \mathbb{R}$, depending only on the unordered frame, with*

$$\Theta_r = \text{Ad}_{a(r)^{it_F}}, \quad \text{i.e.} \quad a(r) \cdot b = a(r)^{1/2+it_F} b a(r)^{1/2-it_F} \text{ for all effects } b,$$

for every $r \in (-\infty, 0]^n$.

Proof. Here $\text{stab}(F)$ is the torus $\{i \text{ diag}(\theta_1, \dots, \theta_n)\} / (\mathbb{R} i \mathbb{1})$ and $\rho_{ij}(d\chi(r)) = i(\theta_i(r) - \theta_j(r))$ on $V_{ij} \cong \mathbb{C}$. Only the differences $\theta_i - \theta_j$ are gauge-invariant (the torus Lie algebra is $\mathbb{R}^n / \mathbb{R}(1, \dots, 1)$), so work with the block weights $\ell_{ij}(r) := \theta_i(r) - \theta_j(r)$ directly, avoiding any choice of representative. Lemma 4.3 gives $\ell_{ij}(r) = (r_i - r_j) t_{ij}$ for reals t_{ij} , and the weights obey the cocycle $\ell_{ik} = \ell_{ij} + \ell_{jk}$. Hence for distinct i, j, k (available since $n \geq 3$),

$$t_{ik}(r_i - r_k) = t_{ij}(r_i - r_j) + t_{jk}(r_j - r_k) \quad \text{for all } r;$$

comparing the coefficients of r_i and of r_k gives $t_{ik} = t_{ij}$ and $t_{ik} = t_{jk}$. So $t_{ij} =: t_F$ is one constant per frame, and $\rho_{ij}(d\chi(r)) = i t_F (r_i - r_j)$ on every block. This constant is independent of the labelling of the atoms of F : reversing an ordered pair sends $\ell_{ij} \mapsto \ell_{ji} = -\ell_{ij}$ and $r_i - r_j \mapsto r_j - r_i = -(r_i - r_j)$, leaving the coefficient t_F in $\ell_{ij}(r) = t_F(r_i - r_j)$ unchanged; since permutations are generated by transpositions, t_F depends only on the unordered frame. So Θ_r acts on V_{ij} by the rotation $e^{it_F(r_i - r_j)}$, i.e. $\Theta_r = \text{Ad}_{a(r)^{it_F}}$ (with $a(r)^{it_F} = \sum_k e^{it_F r_k} p_k$, whose conjugation rotates the (i, j) corner by $e^{it_F(r_i - r_j)}$), and $L_{a(r)} = Q_{\sqrt{a(r)}} \text{Ad}_{a(r)^{it_F}}$, i.e. $a(r) \cdot b = a(r)^{1/2+it_F} b a(r)^{1/2-it_F}$. $\square$

Lemma 5.6 (Adjacent-frame constancy). *Let $n \geq 3$, and let F and F' be adjacent Jordan frames of $M_n(\mathbb{C})^{\text{sa}}$: frames differing by a rotation inside a rank-two block $q = p_1 + p_2 = p'_1 + p'_2$ and sharing the atoms $p_3, \dots, p_n$. Then $t_F = t_{F'}$.*

Proof. Fix $k \geq 3$ and let $X = \{x : q \circ x = \frac{1}{2}x = p_k \circ x\}$ be the cross-coherence space, so $X = V_{1k} \oplus V_{2k} = V'_{1k} \oplus V'_{2k}$ in the two frames. Take $a = \lambda q + \sum_{l \geq 3} \lambda_l p_l$ with $\lambda, \lambda_l \in (0, 1)$ distinct. By Lemma 4.2 Θ_a fixes $J_2(q)$ pointwise, and Proposition 5.5 gives $\Theta_a = \text{Ad}_{a^{it_F}}$ with a^{it_F} commuting with q and p_k ; so Lemma 5.4(iii) yields

$$\Theta_a|_X = \exp(t_F (\log \lambda - \log \lambda_k) \mathcal{J}_{q,k}),$$

the two subblocks V_{1k}, V_{2k} carrying the same phase because a has the repeated eigenvalue λ on $\text{range}(q)$. The oriented generator $\mathcal{J}_{q,k}$ depends only on q, p_k , and the ambient i

(Lemma 5.4(i)), *not* on the splitting $q = p_1 + p_2$; so the same operator Θ_a , computed in F' , acts on X as $\exp(t_{F'}(\log \lambda - \log \lambda_k) \mathcal{J}_{q,k})$ against the *same* generator. The comparison is licensed by two further facts. First, $\Theta_a \in \text{Stab}(F)^\circ$ is *inner*—the identity component of $\text{Aut}(M_n(\mathbb{C})^{\text{sa}})$ acts by $X \mapsto UXU^*$ (Proposition 4.4)—hence $\mathbb{C}$ -linear on X and commuting with $\mathcal{J}_{q,k}$ (Lemma 5.4(ii)); the orientation-reversing transpose, which would reverse $\mathcal{J}_{q,k}$, is excluded. Second, the identification is independent of the unitary matrix units aligning a general frame with the standard one: such a change conjugates $\mathcal{J}_{q,k}$ by a unitary fixing q and p_k , under which it is invariant (Lemma 5.4(ii)). Writing $s = \log \lambda - \log \lambda_k$ and letting λ vary over an open interval, s ranges over an interval and $\exp(t_F s \mathcal{J}_{q,k}) = \exp(t_{F'} s \mathcal{J}_{q,k})$ for all such s . Differentiating at an interior point and cancelling the common invertible exponential gives $t_F \mathcal{J}_{q,k} = t_{F'} \mathcal{J}_{q,k}$; since $\mathcal{J}_{q,k}^2 = -\mathbb{1}_X$ (so $\mathcal{J}_{q,k} \neq 0$), this forces $t_F = t_{F'}$ *exactly* (not $t_F = -t_{F'}$), with no 2π ambiguity. $\square$

Theorem 5.7 (Complex type). *On $H_n(\mathbb{C}) = M_n(\mathbb{C})^{\text{sa}}$, $n \geq 3$: there is a single $t \in \mathbb{R}$ with $a \cdot b = a^{1/2+it} b a^{1/2-it}$ for all effects a, b .*

Proof. Each frame carries one parameter (Proposition 5.5), adjacent frames carry the same one (Lemma 5.6), and any two frames are joined by finitely many adjacent moves (Lemma 5.8), so a single global t results. Every invertible effect is diagonal in some frame, so $a \cdot b = a^{1/2+it} b a^{1/2-it}$ for all invertible a ; density (Lemma 5.9, both sides norm-continuous) extends it to all effects. $\square$

Lemma 5.8 (Frame connectivity). *For $n \geq 2$, the graph on Jordan frames of $M_n(\mathbb{C})^{\text{sa}}$ with $F \sim F'$ iff they share $n - 2$ atoms (differ only inside a rank-two projection P) is connected.*

Proof. Frames are orthonormal bases up to phase and ordering. Let $p = ww^*$ be an atom of the target frame and write $w = \sum_i c_i v_i$ in the current orthonormal basis $\{v_i\}$ (so the current frame is $\{v_i v_i^*\}$). If $m > 1$ of the coefficients are nonzero, choose $i \neq j$ with $c_i, c_j \neq 0$, put $\rho = (|c_i|^2 + |c_j|^2)^{1/2} > 0$ and $u = \rho^{-1}(c_i v_i + c_j v_j)$, and complete u to an orthonormal pair $\{u, u^\perp\}$ of $\text{span}\{v_i, v_j\}$. Replacing $\{v_i, v_j\}$ by $\{u, u^\perp\}$ changes exactly two atoms of the frame, one edge of the graph. In the new basis w has component $\rho \neq 0$ along u and 0 along $u^\perp$: its remaining coordinates are orthogonal to $u^\perp \in \text{span}\{v_i, v_j\}$, and $\langle u^\perp, w \rangle = \rho \langle u^\perp, u \rangle = 0$. So the number of nonzero coordinates of w drops by one. After $\leq n - 1$ edges p is an atom of the current frame; all subsequent moves may be taken inside $p^\perp$, and induction on n installs the remaining target atoms, the base case $n = 2$ being a single move. $\square$

Lemma 5.9 (Singular extension). *In each of Proposition 5.1 and Theorems 5.2, 5.3 and 5.7, the stated formula extends from invertible to all effects by (S2)-continuity.*

Proof. For any effect a , $a_\varepsilon = (1 - \varepsilon)a + \varepsilon e$ is invertible for $\varepsilon \in (0, 1]$ and $a_\varepsilon \rightarrow a$; (S2) gives $L_{a_\varepsilon}(b) \rightarrow L_a(b)$, and the Jordan functional calculus makes $Q_{\sqrt{a_\varepsilon}} b$ (resp. $a_\varepsilon^{1/2 \pm it}$, with value 0 at eigenvalue 0) continuous, so the invertible-case equality passes to the limit. $\square$

Proof of Theorem 1. Combine Proposition 5.1 (real), Theorem 5.2 (quaternionic), Theorem 5.3 (exceptional), and Theorem 5.7 (complex), with Lemma 5.9. The four coordinate types exhaust the simple Euclidean Jordan algebras of rank ≥ 3 . For the real, quaternionic, and exceptional types the per-frame conclusion $\Theta_r = \text{id}$ is already global, because each invertible effect is diagonal in its own frame, so no frame-connectivity step is needed. Only

the complex type, whose per-frame parameter t_F survives, requires the frame-overlap globalization of Theorem 5.7. The three typewise kill mechanisms—stabilizer discreteness (real), two-slot separation (quaternionic), and block-representation faithfulness (exceptional)—are isolated in the discussion following Proposition 4.4. $\square$

Lemma 5.10 (Sufficiency of the twisted products). *For every $n \geq 2$ and every $t \in \mathbb{R}$, the twisted product $\circ_t$ of (1) is a norm-continuous (S1)–(S7) sequential product on the effects of $H_n(\mathbb{C})$.*

Proof. Set $g(\lambda) = \lambda^{1/2+it}$ with $g(0) := 0$, so that $|g(\lambda)|^2 = \lambda$ and g is multiplicative on $[0, \infty)$: $g(\lambda\mu) = g(\lambda)g(\mu)$. Then $a \cdot b = a^{1/2+it} b a^{1/2-it} = f_a(a) b \bar{f}_a(a)$ with $f_a = g|_{\text{sp}(a)}$ is the functional-calculus product of this character family. Shen–Wu’s construction criterion [SW09, Thm. 3.1] requires of the family $\{f_a\}$ exactly two conditions: (a) $|f_a(\lambda)| = \sqrt{\lambda}$ on the spectrum of a , and (b) for commuting a, b , $f_a(a) f_b(b) = \zeta f_{ab}(ab)$ for some unimodular scalar ζ . Here (a) is $|g(\lambda)|^2 = \lambda$, and (b) holds with phase $\zeta \equiv 1$: commuting positive matrices are jointly diagonalizable, and entrywise $g(\alpha\beta) = g(\alpha)g(\beta)$ (with $g(0) = 0$ covering zero eigenvalues), so $g(a)g(b) = g(ab)$ exactly. Hence the algebraic axioms (S1) and (S3)–(S7) hold [SW09, Thm. 3.1, Ex. 3.1]. For first-variable norm-continuity (S2): g is continuous on $[0, \infty)$ (since $|g(\lambda)| = \lambda^{1/2} \rightarrow 0$), so $a \mapsto a^{1/2+it}$ is norm-continuous on the positive cone by continuity of the functional calculus, and therefore so is $a \mapsto a \cdot b$ for each fixed b . All seven axioms hold, for every real t . $\square$

Remark 5.11 (The labelled classification space of products). For a fixed finite-dimensional EJA J , let $\mathcal{M}_{\text{SP}}(J)$ denote the set of norm-continuous (S1)–(S7) sequential products on its fixed (labelled) effect algebra—a parameter set *before* any quotient by Jordan automorphisms and with no topology imposed, so the identifications below are bijections of sets. Theorem 1 computes it at rank $n \geq 3$:

$$\mathcal{M}_{\text{SP}}(J) = \{\bullet\} \text{ for } M_n(\mathbb{R})^{\text{sa}}, M_n(\mathbb{H})^{\text{sa}}, H_3(\mathbb{O}), \quad \mathcal{M}_{\text{SP}}(H_n(\mathbb{C})) \cong \mathbb{R}, \quad t \mapsto \circ_t,$$

with $\{\bullet\}$ the Lüders singleton and $\circ_t$ the twisted product (1). For the bijection $t \mapsto \circ_t$, well-definedness is Lemma 5.10, surjectivity is Theorem 1, and injectivity follows from Remark 6.10, whose 2×2 interferometric computation embeds in a coherence corner of $H_n(\mathbb{C})$ and there separates every real t . Peirce exchange covariance (Corollary 7.1) selects the point $t = 0$; quotienting by all Jordan automorphisms identifies t with $-t$ through the transpose. At rank two the complex qubit instead carries $\mathcal{M}_{\text{SP}}(M_2(\mathbb{C})^{\text{sa}}) \cong C(\mathbb{RP}^2, \mathbb{R})$ (Corollary 6.8), into which the constant twists embed.

6 The Rank-Two Boundary

At rank two, the preceding argument does not impose cross-frame constancy. Throughout this section, write rank-one projections of $M_2(\mathbb{C})^{\text{sa}}$ as $P_n = \frac{1}{2}(\mathbb{1} + n \cdot \sigma)$ for $n \in S^2$ (σ the Pauli vector; here n is the Bloch vector, not the rank), so $P_{-n} = \mathbb{1} - P_n$. The unordered frames $\{P_n, P_{-n}\}$ are the points $[n]$ of $S^2/\{\pm 1\} = \mathbb{RP}^2$, and $\text{fr } a \in \mathbb{RP}^2$ denotes the frame of a nonscalar effect a .

Ordered frames and their orientation. A point $n \in S^2$ is the same datum as an *ordered* frame (P_n, P_{-n}) , the antipodal map $n \mapsto -n$ reversing the order. Write $W_n := \{x : P_n \circ x = \frac{1}{2}x = P_{-n} \circ x\} \cong \mathbb{R}^2$ for the Peirce coherence block (3) of that frame; inside the traceless self-adjoint 3-space $Y \cong \mathbb{R}^3$ it is the plane $n^\perp$. Orient W_n by

$$\mathcal{J}_n(x) := i P_n x P_{-n} - i P_{-n} x P_n. \quad (5)$$

Writing $x \in W_n$ in the $\{P_n, P_{-n}\}$ eigenbasis as the off-diagonal $\begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix}$ gives $\mathcal{J}_n(x) = \begin{pmatrix} 0 & iz \\ -i\bar{z} & 0 \end{pmatrix}$, so $\mathcal{J}_n$ maps W_n to W_n with $\mathcal{J}_n^2 = -\mathbf{1}_{W_n}$: the orientation is anchored to the ambient i on the coherence corner and invokes no right-hand rule on n . Since $n \mapsto -n$ swaps $P_n \leftrightarrow P_{-n}$,

$$\mathcal{J}_{-n} = -\mathcal{J}_n : \quad (6)$$

reversing the order of a frame reverses the orientation of its block. Formula (5) applied on all of Y annihilates the diagonal $\mathbb{R}n$ (there $P_n x P_{-n} = 0$), so $\mathcal{J}_n$ extends to Y with $\mathcal{J}_n^2 = -\Pi_{W_n}$, minus the orthogonal projection of Y onto W_n , so that $\mathcal{J}_n^2$ has trace -2 on Y . Since P_n depends continuously, indeed real-analytically, on $n \in S^2$, so does $\mathcal{J}_n$ as a real-linear operator on Y .

Proposition 6.1 (Necessity at rank two). *Let $(\cdot)$ be a norm-continuous (S1)–(S7) sequential product on $M_2(\mathbb{C})^{\text{sa}}$. For each ordered frame (P_n, P_{-n}) , $n \in S^2$, there is a single real $\tilde{t}(n)$ such that every invertible effect $a = \lambda_+ P_n + \lambda_- P_{-n}$ diagonal in that frame has comparison map*

$$\Theta_a|_{W_n} = \exp(\ell \tilde{t}(n) \mathcal{J}_n), \quad \ell := \log(\lambda_+/\lambda_-), \quad (7)$$

the log-ratio being taken in the given order and allowed either sign: one real parameter per ordered frame, independent of which effect diagonal in that frame is used. No cross-frame constancy follows; that $\tilde{t}$ depends only on the unordered frame, and so descends to $\mathbb{RP}^2$, is Lemma 6.7, proved below. Singular effects are covered only by continuity (by the ϵ -regularization argument of Lemma 5.9, carried out for this rank-two case in the proof of Corollary 6.8), the ratio λ_+/λ_- being undefined at a zero eigenvalue.

Proof. By Proposition 3.10 (valid at any rank), Θ_a is a Jordan automorphism fixing P_n and P_{-n} . By Lemma 4.1 it lies in the identity component of the frame stabilizer, so it acts on the single block W_n as a rotation in $\text{SO}(2)$, not a reflection: in the orientation (5), as $\exp(\theta \mathcal{J}_n)$ for a real angle θ . For diagonal $a(r) = e^{r_1} P_n + e^{r_2} P_{-n}$ the standard product satisfies $a(r) \bullet a(r') = a(r + r')$, so Proposition 5.7 of [vdW18] (Proposition 3.10) makes $r \mapsto \Theta_{a(r)}|_{W_n}$ a continuous homomorphism on the negative orthant, extended to a homomorphism $\mathbb{R}^2 \rightarrow \text{SO}(2)$ by the cancellation extension of Lemma 4.3 (valid at any rank). By coalescence (Lemma 4.2, valid at any rank) it is trivial when $r_1 = r_2$. Since $\mathbb{R}^2$ is simply connected, a continuous homomorphism $\mathbb{R}^2 \rightarrow \text{SO}(2)$ lifts through the universal cover $\mathbb{R} \rightarrow \text{SO}(2)$ to a continuous homomorphism $\mathbb{R}^2 \rightarrow \mathbb{R}$, i.e. to a linear functional. One killing the diagonal is a multiple of $r_1 - r_2$, so the homomorphism is $r \mapsto \exp((r_1 - r_2) \tilde{t}(n) \mathcal{J}_n)$ for a single real $\tilde{t}(n)$. Here $r_1 - r_2$ is exactly the ordered log-ratio ℓ , which gives (7) for either sign of ℓ . No magnitude ordering of the two eigenvalues is imposed anywhere in this argument: the parameter is attached to the ordered frame, and the sign of ℓ is carried by the formula. Distinct frames of $M_2(\mathbb{C})^{\text{sa}}$ share no atom and the complement of a rank-one P is rank one, so there is no third projection to compare frames. The frame-overlap step of Lemma 5.6, which acts on the cross-coherence space of a shared spectator atom, does not apply. $\square$

In fact the rank-two products are completely classified. We prove the constructive half first: every continuous frame function yields a product (Proposition 6.3), with an explicit frame-dependent family witnessing that the freedom is real (Theorem 6.4). The converse then closes: the frame parameter of Proposition 6.1 is bounded (Lemma 6.5), continuous (Lemma 6.6), and descends to $\mathbb{RP}^2$ (Lemma 6.7), so the norm-continuous (S1)–(S7) products on $M_2(\mathbb{C})^{\text{sa}}$ are exactly $C(\mathbb{RP}^2, \mathbb{R})$ (Corollary 6.8).

Remark 6.2 (Rank-two selection consequences). Two selector consequences follow for the two rank-two qubits without any exhaustive classification. First, Proposition 5.1 applies at rank two: the real qubit’s unique norm-continuous (S1)–(S7) product is the Lüders product, whose coherence-block action $\sqrt{xy} \text{Id}$ is Peirce exchange covariant. Second, for $M_2(\mathbb{C})^{\text{sa}}$: by Proposition 6.1 and Lemma 4.1 the block action of an invertible nonscalar $a = \lambda_+ P_n + \lambda_- P_{-n}$ is $L_a|_{W_n} = Q_{\sqrt{a}} \Theta_a|_{W_n} = \sqrt{\lambda_+ \lambda_-} \exp(\ell \tilde{t}(n) \mathcal{J}_n)$ with $\ell = \log(\lambda_+/\lambda_-)$, the scalar factor being exchange-symmetric. Here Peirce exchange covariance (Section 7) is the fixed-orientation condition: for the fixed ordered frame (P_n, P_{-n}) , exchanging the two eigenvalues—which replaces ℓ by $-\ell$ —leaves unchanged the induced endomorphism of the *same* coherence space W_n , in its fixed ambient orientation $\mathcal{J}_n$. (Relabeling the frame instead would also send $\mathcal{J}_n$ to $\mathcal{J}_{-n} = -\mathcal{J}_n$ by (6) and make the two sign changes cancel; the condition constrains the block action at fixed orientation.) It forces

$$\exp(\ell \tilde{t}(n) \mathcal{J}_n) = \exp(-\ell \tilde{t}(n) \mathcal{J}_n)$$

for ℓ over an interval, hence $\tilde{t}(n) = 0$ for every frame; with scalar effects Lüders and singular effects covered by (S2)-continuity, every norm-continuous (S1)–(S7) product on $M_2(\mathbb{C})^{\text{sa}}$ satisfying Peirce exchange covariance is the Lüders product, the products that fail it being left to the full classification of Corollary 6.8.

Proposition 6.3 (Sufficiency: every continuous frame function is a product). *For every continuous $t : \mathbb{RP}^2 \rightarrow \mathbb{R}$, the operation*

$$a \cdot b := a^{1/2+it_a} b a^{1/2-it_a}, \quad t_a := t(\text{fr } a) \text{ for nonscalar } a, \quad t_a := 0 \text{ for scalar } a$$

is a norm-continuous (S1)–(S7) sequential product on $M_2(\mathbb{C})^{\text{sa}}$; we write $\circ_t$ for it, extending the constant-twist notation of (1).

The algebraic content of this family is Shen–Wu’s Example 3.2 [SW09], which assigns an arbitrary real parameter to each unordered frame. This paper adds (S2) (Step 3), the axiom that on the necessity side trades their arbitrary frame parameter for a continuous one, together with a self-contained verification of all seven axioms.

Proof. Throughout, $F_a := a^{1/2+it_a}$, a normal matrix with $F_a F_a^* = a$, so that uniformly $a \cdot b = F_a b F_a^*$ (for scalar a , $F_a = \sqrt{a}$). The steps use only that t_a is a real number attached to the unordered frame of a , bounded and continuous in the frame, and they are self-contained. The displayed algebraic identities, with the twist value carried as a free real parameter, are additionally corroborated in exact symbolic arithmetic by `verify_n2.py` (labels V1–V8; V9, particular to the family of Theorem 6.4, appears in the same script; see the Supplementary Material).

Step 0. For scalar $a = \lambda \mathbb{1}$, $a \cdot b = \lambda b$ (V2); a nonscalar effect has two distinct eigenvalues and a unique unordered frame $\{P, \mathbb{1} - P\}$, so t_a is well defined.

Step 1 (block form). For nonscalar $a = \lambda_1 P + \lambda_2 P^\perp$, $t = t_a$,

$$a \cdot b = \lambda_1 P b P + \lambda_2 P^\perp b P^\perp + \sqrt{\lambda_1 \lambda_2} (e^{it \log(\lambda_1/\lambda_2)} P b P^\perp + e^{-it \log(\lambda_1/\lambda_2)} P^\perp b P)$$

(V1), the Lüders value for $\lambda_2 = 0$ (V3). Cone-preservation: $b \mapsto F_a b F_a^*$ is positive and $b \leq \mathbb{1}$ gives $F_a b F_a^* \leq a \leq \mathbb{1}$.

Step 2 (S1, S3, S4). (S1): $b \mapsto F_a b F_a^*$ is linear. (S3): $\mathbb{1}$ is scalar with $\lambda = 1$ (Step 0). (S4): suppose $a \cdot b = (F_a \sqrt{b})(F_a \sqrt{b})^* = 0$, so $F_a \sqrt{b} = 0$. With the partial isometry $u = \sum_{\lambda_j > 0} \lambda_j^{-it_a} p_j$ one has $\sqrt{a} = u F_a$, so $\sqrt{a} \sqrt{b} = u F_a \sqrt{b} = 0$; taking adjoints, $\sqrt{b} \sqrt{a} = 0$. Writing $F_b = v \sqrt{b}$ with $v = \sum_{\mu_j > 0} \mu_j^{it_b} q_j$ for the spectral data $b = \sum_j \mu_j q_j$ (note the opposite sign of exponent from u), we get $F_b \sqrt{a} = v \sqrt{b} \sqrt{a} = 0$, whence $b \cdot a = (F_b \sqrt{a})(F_b \sqrt{a})^* = 0$ (instances: V8).

Step 3 (S2: continuity of $a \mapsto a \cdot b$). At a nonscalar a_0 : the eigenvalues are continuous, the frame $P(a) = (a - \lambda_2 \mathbb{1})/(\lambda_1 - \lambda_2)$ and with it $\text{fr } a \in \mathbb{RP}^2$ are continuous on the nonscalar locus, so $t_a = t(\text{fr } a)$ is continuous there. The map $(z, s) \mapsto z^{1/2+is}$ (with value 0 at $z = 0$) is jointly continuous on $[0, 1] \times \mathbb{R}$, since its modulus is $\sqrt{z}$. Hence $a \mapsto F_a$, and with it $a \mapsto a \cdot b$, is continuous at a_0 , including rank-deficient a_0 . At a scalar $a_0 = \lambda_0 \mathbb{1}$: for nonscalar a with eigenvalues λ_1, λ_2 , frame P , twist value t_a , expanding b into its four P -blocks (each blocking map a contraction),

$$\|a \cdot b - \lambda_0 b\| \leq (|\lambda_1 - \lambda_0| + |\lambda_2 - \lambda_0| + 2|\sqrt{\lambda_1 \lambda_2} e^{i\theta} - \lambda_0|) \|b\|, \quad \theta = t_a \log(\frac{\lambda_1}{\lambda_2}),$$

the cross coefficient read as its actual value 0 when $\lambda_2 = 0$. If $\lambda_0 > 0$, then eventually $\lambda_2 > 0$ along $a \rightarrow \lambda_0 \mathbb{1}$ and $\lambda_1, \lambda_2 \rightarrow \lambda_0$, while $|t_a| \leq \|t\|_\infty < \infty$ (t is continuous on the compact $\mathbb{RP}^2$), so $\theta \rightarrow 0$ and every term vanishes. If $\lambda_0 = 0$: since $b \leq \mathbb{1}$, $0 \leq a \cdot b = F_a b F_a^* \leq F_a F_a^* = a$, so $\|a \cdot b\| \leq \|a\| = \max(\lambda_1, \lambda_2) \rightarrow 0$ (using no bound on t). Either way $a \cdot b \rightarrow \lambda_0 b$.

Step 4 (compatibility equals commutation). We show $a \cdot b = b \cdot a \iff ab = ba$. ($\Leftarrow$) If $a = \lambda \mathbb{1}$ is scalar, $a \cdot b = \lambda b$ (Step 0) and $b \cdot a = F_b(\lambda \mathbb{1})F_b^* = \lambda F_b F_b^* = \lambda b$; symmetrically if b is scalar. Otherwise a, b share their unique frame, F_a is a functional-calculus function of a , so $[F_a, b] = 0$ and $a \cdot b = F_a F_a^* b = ab = ba = b \cdot a$. ($\Rightarrow$) Suppose $ab \neq ba$; then a, b are both nonscalar. Since $F_a^* a F_a = a F_a^* F_a = a^2$ (functional calculus) and the trace is cyclic, $\text{tr}((a \cdot b)a) = \text{tr}(F_a b F_a^* a) = \text{tr}(b F_a^* a F_a) = \text{tr}(a^2 b)$. Computing the other pairing in b 's eigenbasis, with $b = \mu_1 Q + \mu_2 Q^\perp$ ($\mu_1 > \mu_2 \geq 0$), $\hat{a}$ the matrix of a there, and $\theta_b = t_b \log(\mu_1/\mu_2)$,

$$\text{tr}(a^2 b) - \text{tr}((b \cdot a)a) = (\mu_1 + \mu_2 - 2\sqrt{\mu_1 \mu_2} \cos \theta_b) |\hat{a}_{12}|^2$$

(and $= \mu_1 |\hat{a}_{12}|^2$ if $\mu_2 = 0$). Here $\mu_1 + \mu_2 - 2\sqrt{\mu_1 \mu_2} \cos \theta_b = (\sqrt{\mu_1} - \sqrt{\mu_2})^2 + 2\sqrt{\mu_1 \mu_2} (1 - \cos \theta_b) > 0$ for $\mu_1 \neq \mu_2$, and $\hat{a}_{12} \neq 0$ exactly when $ab \neq ba$, so the two pairings differ and $a \cdot b \neq b \cdot a$. (The displayed trace identities are corroborated symbolically as V4.)

Step 5 (S6, S7). Compatibility is commutation (Step 4); $[a, b] = 0$ gives $[a, \mathbb{1} - b] = [a, b + c] = 0$ and, as F_b is a function of b , $[a, b \cdot c] = F_b[a, c]F_b^* = 0$.

Step 6 (S5; the phase cocycle). Let $a \mid b$, so $ab = ba$ and $a \cdot b = ab$ (Step 4). We claim $F_a F_b = \zeta(a, b) F_{ab}$ with $|\zeta(a, b)| = 1$, from which (S5) follows: $a \cdot (b \cdot c) = F_a F_b c F_b^* F_a^* =$

$F_{ab} c F_{ab}^* = (ab) \cdot c = (a \cdot b) \cdot c$. Since a, b commute they are jointly diagonalizable, with eigenvalue pairs (α_k, β_k) . If a, b, ab are all nonscalar they share one frame, hence one twist value t , and entrywise $\alpha_k^{1/2+it} \beta_k^{1/2+it} = (\alpha_k \beta_k)^{1/2+it}$ (exact power laws for nonnegative reals, with $0 = 0$ at a zero eigenvalue), so $\zeta = 1$. If both are scalar, $\zeta = 1$. If $b = \beta \mathbb{1}$ is scalar ($\beta > 0$) and a nonscalar, then βa is nonscalar with the same frame, $t_{\beta a} = t_a =: s$ (the frames coincide), and $F_a F_b = \sqrt{\beta} F_a = \beta^{-is} F_{\beta a}$, so $\zeta = \beta^{-is}$; if $\beta = 0$ both sides vanish, and symmetrically for a scalar. If a, b are nonscalar but $ab = \gamma \mathbb{1}$ is scalar with $\gamma > 0$, then $\log(\alpha_1/\alpha_2) = -\log(\beta_1/\beta_2)$ and $F_a F_b = \sqrt{\gamma} e^{it \log \gamma} \mathbb{1} = e^{it \log \gamma} F_{ab}$, so $\zeta = e^{it \log \gamma}$; if $\gamma = 0$ both sides vanish. In every case ζ may be taken with $|\zeta| = 1$. (The case verification is corroborated symbolically as V5–V6.) $\square$

Theorem 6.4 (The rank-two boundary: rigidity genuinely fails). *Let $V = M_2(\mathbb{C})^{\text{sa}}$. Fix a rank-one projection R and, for a nonscalar effect a with spectral frame $F = \{P, \mathbb{1} - P\}$, set*

$$\tau(a) := (2 \operatorname{tr}(PR) - 1)^2 \in [0, 1], \quad (8)$$

a well-defined function of the frame; $\tau(a) := 0$ for scalar a . Define

$$a \cdot b := a^{1/2+i\tau(a)} b a^{1/2-i\tau(a)}, \quad 0^z := 0. \quad (9)$$

Then:

- (i) *the operation (9) maps effects to effects; on the ordered frame (P_n, P_{-n}) of a nonscalar effect $a = \lambda_+ P_n + \lambda_- P_{-n}$ its coherence action on W_n is the rotation*

$$\sqrt{\lambda_+ \lambda_-} \exp(\ell \tau(a) \mathcal{J}_n), \quad \ell = \log(\lambda_+ / \lambda_-)$$

(for $\lambda_- > 0$; at $\lambda_- = 0$ the prefactor $\sqrt{\lambda_+ \lambda_-}$ vanishes and, by $0^z := 0$, the coherence action is the Lüders value 0);

- (ii) *it satisfies all seven axioms (S1)–(S7);*
- (iii) *it is genuinely frame-dependent: there is no pair (Φ, t) , consisting of a Jordan automorphism Φ of V and a constant $t \in \mathbb{R}$, such that $a \cdot b = \Phi^{-1}(\Phi(a) \circ_t \Phi(b))$ for all effects. In particular the complex conclusion of Theorem 1 fails at rank two.*

Proof. τ is a continuous frame function. Replacing P by $\mathbb{1} - P$ negates $2 \operatorname{tr}(PR) - 1$, which the square in (8) undoes, so τ is a well-defined function of the unordered frame. As a function of $[n] \in \mathbb{RP}^2$ it is continuous, being a polynomial in $\operatorname{tr}(P_n R)$ with P_n continuous in n and the value even under $n \mapsto -n$. For scalar a both (9) and the operation of Proposition 6.3 give the Lüders value, so the operation (9) is the operation of Proposition 6.3 at the frame function τ . (ii) follows from Proposition 6.3, and Step 1 there gives (i): the block form, the coherence rotation with twist value $\tau(a)$, the Lüders value at $\lambda_- = 0$ (V3), and effect preservation, on taking $P = P_n$ and $(\lambda_1, \lambda_2) = (\lambda_+, \lambda_-)$ there.

Frame dependence (iii). $\tau = 1$ on the frame of R and 0 on any frame with $\operatorname{tr}(PR) = \frac{1}{2}$ (V9). The unital Jordan automorphisms of $M_2(\mathbb{C})^{\text{sa}}$ are $X \mapsto UXU^*$ and $X \mapsto UX^\top U^*$

$(U \in SU(2))$,³ acting on the Bloch ball as $SO(3)$ and its orientation-reversing coset, i.e. as $O(3)$; the conjugations preserve the block orientation $\mathcal{J}_n$ (hence t) and the transpose reverses it (hence $t \mapsto -t$). So any Jordan automorphism Φ and constant t transport $\circ_t$ to a product whose twist is $\pm t$ on every frame, constant in magnitude; since τ takes both values 0 and 1, no (Φ, t) reproduces (9). $\square$

Lemma 6.5 (The frame parameter is bounded). *For any norm-continuous (S1)–(S7) product on $M_2(\mathbb{C})^{\text{sa}}$, $M := \sup_{n \in S^2} |\tilde{t}(n)| < \infty$, with $\tilde{t}$ the ordered-frame parameter of Proposition 6.1.*

Proof. Norms here are the operator norms on the finite-dimensional space of endomorphisms of $M_2(\mathbb{C})^{\text{sa}}$ induced by the order unit norm; all norms on that space are equivalent, so which one is used affects no statement below. By (S2) and finite dimension $a \mapsto L_a$ is operator-norm continuous on invertible effects, and $a \mapsto Q_{a^{-1/2}}$ is continuous, so $a \mapsto \Theta_a = Q_{a^{-1/2}} L_a$ is operator-norm continuous (as in Lemma 4.1). If $|\tilde{t}(n_k)| \rightarrow \infty$ along some sequence $n_k \in S^2$, fix $\lambda \in (0, 1)$, put $s_k = \pi/|\tilde{t}(n_k)| \rightarrow 0$ and $a_k = \lambda e^{s_k/2} P_{n_k} + \lambda e^{-s_k/2} P_{-n_k}$ (an effect for all large k : since $s_k \rightarrow 0$, $\lambda e^{s_k/2} \rightarrow \lambda < 1$, so the finitely many terms with $\lambda e^{s_k/2} > 1$ are discarded, and the contradiction uses only the tail). Then $a_k = \lambda \mathbb{1} + O(s_k) \rightarrow \lambda \mathbb{1}$ uniformly in the frame (the perturbation is $O(s_k)$ and $\|P_{n_k}\| = 1$), so $\Theta_{a_k} \rightarrow \Theta_{\lambda \mathbb{1}} = \mathbb{1}$. (Here $\Theta_{\lambda \mathbb{1}} = \mathbb{1}$ because $\lambda \mathbb{1}$ operator-commutes with every effect, so Proposition 3.10 fixes the whole algebra.) But a_k has ordered log-ratio s_k in the frame n_k , so by (7) its rotation angle on W_{n_k} is $\tilde{t}(n_k)s_k = \pm\pi$, giving $\Theta_{a_k}|_{W_{n_k}} = -\mathbb{1}$. Any norm-one $x_k \in W_{n_k}$ then has $(\Theta_{a_k} - \mathbb{1})x_k = -2x_k$, so $\|\Theta_{a_k} - \mathbb{1}\| \geq 2$ even though the plane W_{n_k} varies with k , a contradiction. Hence $\sup_n |\tilde{t}(n)| < \infty$. $\square$

Lemma 6.6 (Continuity of the ordered-frame parameter). *The ordered-frame parameter $\tilde{t}$ of Proposition 6.1 is continuous on S^2 .*

Proof. Fix $\lambda \in (0, 1)$ and $s > 0$ with $sM < \pi$ (M finite by Lemma 6.5) and $\lambda e^{s/2} \leq 1$, and set $a_n(s) = \lambda e^{s/2} P_n + \lambda e^{-s/2} P_{-n}$, an effect with ordered log-ratio s , continuous in n . Work on the *fixed* traceless self-adjoint 3-space Y . By Proposition 6.1 and Lemma 4.1, $\Theta_{a_n(s)}|_Y = \mathbb{1}_{\mathbb{R}n} \oplus (\text{rotation of } W_n) \in SO(3)$ with rotation angle $s\tilde{t}(n)$, $|s\tilde{t}(n)| \leq sM < \pi$. It therefore has no eigenvalue -1 , and its principal logarithm is defined and is a continuous $\mathfrak{so}(3)$ -valued function of n ,

$$K_n := \text{Log}(\Theta_{a_n(s)}|_Y) = s\tilde{t}(n) \mathcal{J}_n$$

(both sides annihilate $\mathbb{R}n$ and act as the angle- $s\tilde{t}(n)$ generator on W_n). Recover the scalar by the explicit formula

$$\tilde{t}(n) = -\frac{1}{2s} \text{tr}_Y(\mathcal{J}_n K_n),$$

valid because $\text{tr}_Y(\mathcal{J}_n K_n) = s\tilde{t}(n) \text{tr}_Y(\mathcal{J}_n^2) = s\tilde{t}(n) \cdot (-2)$, since $\mathcal{J}_n^2 = -\Pi_{W_n}$ has trace -2 on Y (killing the n -axis). As $n \mapsto \mathcal{J}_n$ and $n \mapsto K_n$ are continuous, $\tilde{t}$ is continuous on S^2 ; the trace is now taken over the fixed space Y , not over a moving 2-plane. $\square$

³A unital Jordan automorphism Φ of $M_2(\mathbb{C})^{\text{sa}}$ fixes the trace, and since $(x \cdot \sigma)^2 = |x|^2 I$ for the Pauli vector σ it preserves the Euclidean norm on the traceless part, so it acts there as some $O \in O(3)$. The adjoint action of $SU(2)$ realizes all of $SO(3)$ and the transpose realizes the reflection reversing the σ_y -axis, giving the two stated forms.

Lemma 6.7 (Evenness and descent to $\mathbb{RP}^2$). $\tilde{t}(-n) = \tilde{t}(n)$ for all $n \in S^2$: reversing the order of a frame reverses the orientation of its block (6) and negates the ordered log-ratio, and those two sign changes cancel in (7). The frame parameter therefore descends to a single continuous, bounded function $t : S^2/\{\pm 1\} = \mathbb{RP}^2 \rightarrow \mathbb{R}$ of the unordered frame—a genuine $\mathbb{RP}^2$ function, with no residual line-bundle twist.

Proof. With $a_n(s) = \lambda e^{s/2} P_n + \lambda e^{-s/2} P_{-n}$ as in Lemma 6.6 (the same fixed s , so that $sM < \pi$; both $a_n(\pm s)$ are effects because $\lambda e^{s/2} \leq 1$), the swap $P_n \leftrightarrow P_{-n}$ gives the single effect two descriptions, $a_{-n}(s) = a_n(-s)$, hence one comparison map $\Theta_{a_{-n}(s)} = \Theta_{a_n(-s)}$, and $W_{-n} = W_n$. Read off each description by (7): in the ordered frame $-n$ the ordered log-ratio is $+s$, and in the ordered frame n it is $-s$, so

$$\exp(s\tilde{t}(-n) \mathcal{J}_{-n}) = \exp(-s\tilde{t}(-n) \mathcal{J}_n) \quad \text{must equal} \quad \exp(-s\tilde{t}(n) \mathcal{J}_n),$$

the first equality by the oddness (6) of the orientation. Both exponents have angle in $(-\pi, \pi)$ against the same generator $\mathcal{J}_n$, so equality of the two rotations forces $\tilde{t}(-n) = \tilde{t}(n)$. No eigenvalue-magnitude comparison enters: (7) holds at either sign of the ordered log-ratio, so the two readings of the same effect are comparable. Boundedness is Lemma 6.5, and continuity passes to the quotient. $\square$

Corollary 6.8 (Rank-two classification). *The map $t \mapsto \circ_t$ of Proposition 6.3 is a bijection from $C(\mathbb{RP}^2, \mathbb{R})$ onto the norm-continuous (S1)–(S7) sequential products on $M_2(\mathbb{C})^{\text{sa}}$, an explicit bijection of sets for the fixed labelled complex matrix algebra $M_2(\mathbb{C})^{\text{sa}}$:*

$$\mathcal{M}_{\text{SP}}(M_2(\mathbb{C})^{\text{sa}}) \cong C(\mathbb{RP}^2, \mathbb{R}).$$

Proof. Proposition 6.3 gives the map. For the inverse, any product’s frame parameter is bounded and continuous, and it descends to $\mathbb{RP}^2$ (Proposition 6.1; Lemmas 6.5 to 6.7). The identity $L_a = Q_{\sqrt{a}} \Theta_a$ from Proposition 3.10, the diagonal fixing from Lemma 4.1, and the coherence action from Proposition 6.1 together show that the product equals $\circ_t$ on invertible nonscalar effects. The two sides are matched by taking $P = P_n$ and $(\lambda_1, \lambda_2) = (\lambda_+, \lambda_-)$, so that $\log(\lambda_1/\lambda_2) = \ell$: Step 1 of Proposition 6.3 puts $\sqrt{\lambda_1 \lambda_2} e^{it \log(\lambda_1/\lambda_2)}$ on the $PbP^\perp$ corner, while $L_a = Q_{\sqrt{a}} \Theta_a$ with (7) puts $\sqrt{\lambda_+ \lambda_-} e^{i\ell \tilde{t}(n)}$ there—the same orientation $\mathcal{J}_n$ and the same sign of ℓ on both sides, so $t = \tilde{t}$. Since both sides are (S2)-continuous (Proposition 6.3) and the invertible nonscalar effects are dense, they agree on all effects. Injectivity: if $t \neq t'$ they differ at some frame $F = [n]$; choose $s > 0$ small enough that $0 < s|t(F) - t'(F)| < 2\pi$, and take the nonscalar effect a diagonal in F with eigenvalues in $(0, 1)$ and ordered log-ratio $\ell = s$ in the ordering (P_n, P_{-n}) . Then the coherence rotations $\exp(st(F) \mathcal{J}_n)$ and $\exp(st'(F) \mathcal{J}_n)$ are distinct—their angles differ by less than a full turn—so they differ on any b with nonzero $P_n b P_{-n}$ corner, and $\circ_t \neq \circ_{t'}$. The target is the function space $C(\mathbb{RP}^2, \mathbb{R})$, not a twisted-section space: the orientation datum cancels in the frame parameter (Lemma 6.7), so $\tilde{t}$ is a function on $\mathbb{RP}^2$ with no residual bundle obstruction. $\square$

Remark 6.9 (Attribution). The non-Lüders freedom is Shen–Wu’s [SW09, Thm. 3.1, Ex. 3.2] (their frame parameter arbitrary and not required continuous); Theorem 6.4 specializes it to a continuous example verified against the full seven-axiom list, and the new content is the necessity half of Corollary 6.8—see the Liu–Wu/Shen–Wu paragraph of Section 8. At

rank two compatible nonscalar effects share their unique frame, so (S5)–(S7) never compare distinct nonscalar frames.

Remark 6.10 (Operational detectability of the twist). The parameter t in the complex family $a \circ_t b = a^{1/2+it} b a^{1/2-it}$ has an interferometric signature on a single qubit. Take the unsharp first effect $a_r = \text{diag}(e^{-r}, 1)$, the sharp second effect $b = |+\rangle\langle+|$, and prepare $\rho_\phi = \frac{1}{2} \begin{pmatrix} 1 & e^{-i\phi} \\ e^{i\phi} & 1 \end{pmatrix}$. The ordered joint probability that a_r then b both return “yes” is

$$p_t(r, \phi) = \text{tr}(\rho_\phi (a_r \circ_t b)) = \frac{1 + e^{-r}}{4} + \frac{e^{-r/2}}{2} \cos(\phi - tr).$$

Scanning the state phase isolates two quadratures, $p_t(r, 0) \propto \cos(tr)$ and $p_t(r, \frac{\pi}{2}) \propto \sin(tr)$ above the t -independent pedestal $\frac{1}{4}(1 + e^{-r})$; a single unsharp $r > 0$ fixes t modulo $2\pi/r$, and two values with irrational ratio fix $t \in \mathbb{R}$ in the noiseless model. Unsharpness is essential: for a projection a one has $a^{1/2 \pm it} = a$, hence $a \circ_t b = aba$ and the t -dependence vanishes.⁴

7 Selectors: Prior-Art Corollaries

For the complex type the master theorem leaves the one-parameter family; several classical side conditions select $t = 0$. The last two are van de Wetering’s or Gudder–Latréolière’s, in greater generality; we record them as corollaries to locate where selection happens, not as new characterizations. Write $E(x, y)$ for the action of $a \cdot (\cdot)$ on a coherence block V_{ij} when a has eigenvalues x, y on the two atoms p_i, p_j ; by Theorem 5.7 it is $E(x, y) = \sqrt{xy} \exp(t \log(x/y) \mathcal{J})$ with $\mathcal{J}$ the block’s complex structure. Here *Peirce exchange covariance* means $E(x, y) = E(y, x)$ with the block’s complex structure $\mathcal{J}$ held *fixed* (the fixed-orientation condition of Remark 6.2, not a relabelling of p_i, p_j).

Corollary 7.1 (Selectors that fix $t = 0$). *On $H_n(\mathbb{C})$, $n \geq 3$, the complex-type product of Theorem 1 is the Lüders product ($t = 0$) if any of the following holds: (i) Peirce exchange covariance, $E(x, y) = E(y, x)$ on every coherence block; (ii) trace-form symmetry, $\langle a \cdot b, c \rangle = \langle b, a \cdot c \rangle$; (iii) covariance under every unital order automorphism (on $M_n(\mathbb{C})^{\text{sa}}$ the transpose suffices).*

Proof. Write the block action as $\sqrt{xy} \exp(t \log(x/y) \mathcal{J})$. (i) $E(x, y) = E(y, x)$ for all x, y iff $t = 0$. (ii) Trace-orthogonality of Peirce spaces makes symmetry equivalent to self-adjointness of the block rotation $\exp(t \log(x/y) \mathcal{J})$ for all admissible x, y . A real rotation is self-adjoint exactly at angles in $\pi\mathbb{Z}$, but $t \log(x/y)$ ranges over an interval as x, y vary, so this forces $t = 0$. (iii) The transpose acts on each block by $\mathcal{J} \mapsto -\mathcal{J}$, forcing $t = -t$. $\square$

Attribution of the selectors, and the weakenings that fail to select, are recorded with the prior art in Section 8.

⁴This is a tomography calculation for the Liu–Wu family, not new structure. Liu and Wu [LW09] introduce the one-parameter product $A^{1/2+ti} B A^{1/2-ti}$, $t \in \mathbb{R}$ (their Remark 1), and compute its explicit qubit form (their Corollary 4.4, $a, b > 0$ branch, with the t restored in their Remark 1): for $A = \text{diag}(a^2, b^2)$ and $B = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$ the off-diagonal entry carries the phase $\theta = (\ln a^2 - \ln b^2)t$, exactly the phase e^{-itr} behind the $\cos(\phi - tr)$ above; they already read t as a physical “time parameter.” Shen and Wu [SW09] generalise the construction but state no state-dependent detection formula.

8 Prior Art, Novelty, and Provenance

Gudder–Latrémoière (2008). They characterize the Lüders product on Hilbert-space effects from five conditions [GL08], one a trace-duality condition $\text{tr}((a \cdot b)c) = \text{tr}(b(a \cdot c))$, the Hilbert-space form of selector (ii). The twisted products satisfy Gudder–Latrémoière’s other four conditions, including their weak associativity $A \cdot (A \cdot B) = A^2 \cdot B$; this is distinct from the SEA compatible-associativity axiom (S5), which the twists also satisfy (Liu–Wu, Shen–Wu). Theorem 1 classifies without the trace-duality condition.

Liu–Wu, Shen–Wu. The twisted family (1) is Liu–Wu’s [LW09, Thm. 4.3, Cor. 4.4, Rem. 1], an existence result answering a uniqueness question from Gudder’s open-problems list [Gud05] in the negative: they construct the family and read t as a physical time parameter, but do not classify the (S1)–(S7) products. Their Theorem 4.3 and Corollary 4.4 are constructions, and their rank-two form result (their Theorem 3.2) forces a single phase on the coherence corner for each fixed first argument while leaving the dependence of that phase on the argument unconstrained. None of these shows that every such product lies in the constant- t family, and at rank two it does not: the products there are the whole function space $C(\mathbb{RP}^2, \mathbb{R})$ (Corollary 6.8), inside which the constant- t family is a single line. Shen–Wu [SW09, Thm. 3.1, Exs. 3.1–3.2] classify the functional-calculus ansatz under their SEA axioms, which do not include (S2) (Example 3.1 the constant- t family, Example 3.2 arbitrary frame dependence). The constant- t Liu–Wu family is first-argument norm-continuous, whereas Shen–Wu’s rank-two frame-dependent construction permits arbitrary frame choices and need not satisfy (S2). Their forced values are ansatz-free on all commuting pairs (their Theorem 2.2), not merely for sharp first arguments. Their Theorem 3.1 characterizes which functional-calculus families are products but does not show every product has that form; Theorem 1 and Corollary 6.8 obtain the classification with no functional-calculus assumption, so that *every* norm-continuous (S1)–(S7) product is of that form. Theorem 6.4’s continuous example specializes their construction; for the interferometric read-out and two-value identifiability of t (Remark 6.10) we are aware of no counterpart in either paper.

Van de Wetering; the residual-automorphism question. Theorem 2.2 is his structure theorem. His Sections 4–5 [vdW18] supply the comparison machinery (Propositions 4.20, 4.22, 5.2, 5.3, 5.5, 5.7) and three uniqueness theorems, each adding one selector. His EJA appendix supplies the compatibility bridge (Proposition 3.6). On the complex, Hilbert-space type he reduces the residual comparison automorphism to Ad_{u_a} for a bicommutant unitary u_a (his Lemma 5.8, for simple invertible a ; Remark 3.12), but does not classify the resulting dependence $a \mapsto \text{Ad}_{u_a}$ under the bare axioms, and the reduction has no exceptional-type analogue. For the complex type, in finite dimension and rank at least three, Theorem 1 gives this comparison automorphism the form $\text{Ad}_{a^{it}}$ with one global constant; equivalently, the residual unitary is a^{it} up to a scalar phase, invisible under conjugation. At rank two the residual is instead a full continuous phase field, $\mathcal{M}_{\text{SP}}(M_2(\mathbb{C})^{\text{sa}})$ in bijection with $C(\mathbb{RP}^2, \mathbb{R})$ (Corollary 6.8). Within the complex matrix family only the infinite-dimensional case remains open. We know of no prior statement of the typewise classification. Separately, van de Wetering’s thesis records that his general update rule (1.1), the fully general instrument

form, is not compatible with the sequential axioms, *why* being “still not entirely clear” (Remark 2.2.7 [vdW21]). That remark concerns the sequential-product/instrument boundary, not the residual-automorphism classification settled here. (The same remark attributes to Liu–Wu [LW09] the claim that every $B(H)^{\text{sa}}$ update rule satisfying the axioms has the form $b \mapsto u_a \sqrt{a} b \sqrt{a} u_a^*$ with u_a a unitary commuting with a . That fixes the *shape* of the update, and in finite dimension it is his Lemma 5.8 above; it is the dependence $a \mapsto u_a$, not the shape, that Theorem 1 determines.) In a complementary direction, Westerbaan, Westerbeaan, and van de Wetering classify the normal sequential effect algebras themselves into three structural types [WWvdW20]. The present problem instead fixes the effect algebra, a finite-dimensional Euclidean Jordan algebra, and classifies the norm-continuous (S1)–(S7) products it carries, with normality derived (Lemma 3.4) rather than assumed.

Selector attributions; weakenings that fail. Conditions (ii), (iii) of Corollary 7.1 are van de Wetering’s in strictly greater generality: inner-product symmetry selects the standard product on any EJA (his Theorem 5.14), and covariance under all unital order isomorphisms with norm-continuity does too (his Theorems 5.9/5.11). Remark 3.12 discusses the exceptional case in the proof of his Theorem 5.11. On $B(H)$ the trace-duality route is Gudder–Latréolière’s [GL08]. Three weakenings fail to select, with the twisted family as witness. Transition-probability symmetry does not select, because the twisted products have symmetric atomic transition probabilities $\text{tr}(pq)$. Identity-component covariance does not, because the $\text{SO}(2)$ rotations commute with $t\mathcal{J}$; only the orientation-reversing transpose kills t . Closure of the invertibles does not, because the twisted product of invertibles is again invertible. Van de Wetering’s strictly stronger *invertibility preservation* [vdW18, Def. 5.16], which adds the inverse law $(a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$, does select given norm-continuity [vdW18, Thm. 5.19].

Ledger. The full input ledger of imported results and the detailed catalogue of nearest prior results are collected in the Supplementary Material. That catalogue covers van de Wetering’s own Section 5 and uniqueness theorems, the purity/dagger selectors of [WWvdW19], and the Westerbaan–van de Wetering reconstruction [WvdW22], the last of which explicitly records the remaining assert-map uniqueness gap. Two recent lines of work are adjacent but distinct. Jenčová and Pulmannová study spectrality in convex sequential effect algebras [JP23], characterizing it, under further conditions, by monotone σ -completeness of the maximal commutative subalgebras. The finite spectrality that this paper needs comes for free in finite dimension (Lemma 3.3). Gudder studies the *operation-relative* sequential product $a[\mathcal{I}]b$, whose value depends on the operation $\mathcal{I}$ chosen to measure a [Gud23]. The products classified here are the effect-level ones satisfying (S1)–(S7), and on the complex type each is realized by a canonical single-Kraus operation, $\rho \mapsto (U_a^t)^* \sqrt{a} \rho \sqrt{a} U_a^t$ of (2).

Relation to the pair-local ansatz. A restrictive alternative posits that the update is pair-local and decomposition-independent, acting by the compressions on the diagonal Peirce blocks and blockwise on the coherence spaces. Theorem 1 assumes no such thing: Peirce-block-diagonality and the single block generator are conclusions (Lemmas 4.1 and 4.3) drawn from the comparison map. Among (S1)–(S7) products the pair-local ansatz therefore names

a subclass already covered, not an input to the proof.

Niestegge. The symmetric transition probabilities built into the Euclidean-Jordan (self-dual) setting of Theorem 1 are a restriction, not a convention: there exist rank-two generalized spin factors with *asymmetric* transition probabilities—the non-Hilbertian examples over ℓ^p , $1 < p < \infty$, $p \neq 2$, of Niestegge [Nie24, Thm. 7.1, §10]. These lie outside the present setting; we cite them only for that boundary fact.

9 Closing Remarks and Open Questions

Two questions remain. First, the higher spin factors: Proposition 5.1 settles the real qubit and Section 6 classifies the complex one, but the other simple rank-two Jordan algebras—the spin factors $\mathbb{R} \oplus \mathbb{R}^k$ beyond those two, including $H_2(\mathbb{H})$ and $H_2(\mathbb{O})$ —are untreated. Second, infinite dimensions: the argument uses finite-dimensional spectral resolutions and Lie theory on a finite-dimensional stabilizer, and extending to type I factors or JBW-algebras would require new continuity arguments.

Acknowledgments

The author thanks John van de Wetering for making his sequential product theorems publicly available and clearly stated.

Large-language-model tools were used in the preparation of this work, as follows. Claude models (Anthropic; Opus 4.8 and Fable 5) were used for literature search, exploratory mathematical conversations, manuscript drafting, and assistance with the accompanying Lean 4 proof artifact and SymPy scripts. A GPT model (OpenAI; `gpt-5.6-sol`) was used for pre-submission mathematical consistency checking and critical evaluation of draft arguments. No AI system is an author. All definitions, theorem statements, proofs, and attributions are the author’s; all cited sources were independently inspected, and all mathematical statements, proofs, attributions, and computational outputs were verified by the author. No reference was included on the basis of AI-generated output alone. The author assumes full responsibility for the accuracy, originality, and integrity of the work.

Data and code availability

The Lean 4 proof artifact and the SymPy script `verify_n2.py` that corroborate the paper’s computations are deposited in a permanent public repository at the immutable commit `ehrllich-b/research@b7db3e8` (directory `twist-normal-form-lean/`). A Zenodo archival DOI will be minted from the archived source before publication. The Lean development is a self-contained Lake project (Mathlib v4.28.0). Which statements are machine-checked, which are carried as cited interface hypotheses, and which are not formalized at all are recorded in the archive’s theorem-to-file map and in the accompanying Supplementary Material, together with the axiom-closure audit and the SymPy identity list. No other data were generated or analysed in this work.

References

- [AS03] Erik M. Alfsen and Frederic W. Shultz. *Geometry of State Spaces of Operator Algebras*. Birkhäuser, Boston, 2003.
- [Bae02] John C. Baez. The octonions. *Bulletin of the American Mathematical Society*, 39(2):145–205, 2002.
- [BGW20] Howard Barnum, Matthew A. Graydon, and Alexander Wilce. Composites and categories of Euclidean Jordan algebras. *Quantum*, 4:359, 2020. arXiv:1606.09331.
- [FK94] Jacques Faraut and Adam Korányi. *Analysis on Symmetric Cones*. Oxford University Press, 1994.
- [GG02] Stan Gudder and Richard Greechie. Sequential products on effect algebras. *Reports on Mathematical Physics*, 49(1):87–111, 2002.
- [GL08] Stan Gudder and Frédéric Latrémolière. Characterization of the sequential product on quantum effects. *Journal of Mathematical Physics*, 49(5):052106, 2008. arXiv:0803.3867.
- [Gud05] Stan Gudder. Open problems for sequential effect algebras. *International Journal of Theoretical Physics*, 44(12):2199–2206, 2005.
- [Gud23] Stanley Gudder. Properties of sequential products, 2023. arXiv:2307.16327.
- [JP23] Anna Jenčová and Sylvia Pulmannová. Spectrality in convex sequential effect algebras. *International Journal of Theoretical Physics*, 62(8):193, 2023.
- [JvNW34] Pascual Jordan, John von Neumann, and Eugene Wigner. On an algebraic generalization of the quantum mechanical formalism. *Annals of Mathematics*, 35(1):29–64, 1934.
- [Loo77] Ottmar Loos. *Bounded Symmetric Domains and Jordan Pairs*. University of California, Irvine (lecture notes), 1977.
- [LW09] Weihua Liu and Junde Wu. A uniqueness problem of the sequence product on operator effect algebra $\mathcal{E}(H)$. *Journal of Physics A: Mathematical and Theoretical*, 42(18):185206, 2009. arXiv:0812.0630.
- [Nie24] Gerd Niestegge. Non-symmetric transition probability in generalized qubit models. *Foundations of Physics*, 54(1):9, 2024. arXiv:2208.07135.
- [SV00] Tonny A. Springer and Ferdinand D. Veldkamp. *Octonions, Jordan Algebras and Exceptional Groups*. Springer Monographs in Mathematics. Springer, Berlin, Heidelberg, 2000.

- [SW09] Jun Shen and Junde Wu. Sequential product on standard effect algebra $\mathcal{E}(H)$. *Journal of Physics A: Mathematical and Theoretical*, 42(34):345203, 2009. arXiv:0905.0596.
- [vdW18] John van de Wetering. Three characterisations of the sequential product. *Journal of Mathematical Physics*, 59(8):082202, 2018. arXiv:1803.08453.
- [vdW19] John van de Wetering. Sequential product spaces are Jordan algebras. *Journal of Mathematical Physics*, 60(6):062201, 2019. arXiv:1803.11139.
- [vdW20] John van de Wetering. Commutativity in Jordan operator algebras. *Journal of Pure and Applied Algebra*, 224(11):106407, 2020. arXiv:1912.01903.
- [vdW21] John van de Wetering. *Quantum Theory from Principles, Quantum Software from Diagrams*. PhD thesis, Radboud University Nijmegen, 2021. arXiv:2101.03608.
- [vIR20] Hendrik van Imhoff and Mark Roelands. Order isomorphisms between cones of JB-algebras. *Studia Mathematica*, 254:179–198, 2020. arXiv:1904.09278.
- [WvdW22] Bas Westerbaan and John van de Wetering. A computer scientist’s reconstruction of quantum theory. *Journal of Physics A: Mathematical and Theoretical*, 55(38):384002, 2022. arXiv:2109.10707.
- [WWvdW19] Abraham Westerbaan, Bas Westerbaan, and John van de Wetering. Pure maps between Euclidean Jordan algebras. *Electronic Proceedings in Theoretical Computer Science*, 287:345–364, 2019. arXiv:1805.11496.
- [WWvdW20] Abraham Westerbaan, Bas Westerbaan, and John van de Wetering. The three types of normal sequential effect algebras. *Quantum*, 4:378, 2020. arXiv:2004.12749.
- [Yok09] Ichiro Yokota. Exceptional Lie groups, 2009. arXiv:0902.0431, 204 pp. monograph.